The first three exercises in this series are dedicated to the proof of the general Marcinkiewicz interpolation theorem (Theorem 4.3 in the script) given in Appendix B of Stein's book[1].

Exercise 9.1.
In this exercise we will prove the following inequality for a non-increasing function $h : (0,\infty) \to [0,\infty)$:

$$\left(\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{\mathrm{d}t}{t}\right)^{1/q_2} \leq A\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1},$$

where $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ and A is a constant depending on q_1, q_2 and p.

(a) Show this first for $q_2 = \infty$ (where the left hand side is interpreted as the supremum).

(b) Then show it for every $q_1 < q_2 < \infty$.

Exercise 9.2.
Prove the "second Hardy inequality": for a measurable function $f : (0,\infty) \to [0,\infty)$, and numbers $p \geq 1$ and $r > 0$,

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{1/p} \leq \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{r-1}\,\mathrm{d}y\right)^{1/p}.$$

Hint: recall the proof of the first Hardy inequality (Exercise 8.4).

Exercise 9.3.
The goal of this (long) exercise is to prove the general form of the Marcinkiewicz interpolation theorem. Assume we are given exponents

$$1 \leq p_0 \leq q_0 \leq \infty \quad \text{and} \quad 1 \leq p_1 \leq q_1 \leq \infty \quad \text{with} \quad p_0 < p_1 \quad \text{and} \quad q_0 \neq q_1.$$

Let T be a sub-additive operator defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ and assume that T is of weak type (p_i, q_i) for $i = 0, 1$, meaning that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}) \leq \left(\frac{A_i\|f\|_{L^{p_i}}}{\alpha}\right)^{q_i} \qquad \forall \alpha > 0$$

in case $q_i < \infty$, and in case $q_i = \infty$, that $\|Tf\|_{L^\infty} \leq A_i\|f\|_{L^{p_i}}$. The theorem then states that, given $0 < \theta < 1$ and letting

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

[1]Stein, Elias M. *Singular Integrals and Differentiability Properties of Functions*, Princeton: Princeton University Press, 1971. https://doi.org/10.1515/9781400883882

then T is of strong type (p,q), meaning that $\|Tf\|_{L^q} \leq A\|f\|_{L^p}$ for a constant A depending on p_0, p_1, q_0, q_1 and θ.

We define the parameter σ as

$$\sigma = \frac{1/q_0 - 1/q}{1/p_0 - 1/p} = \frac{1/q - 1/q_1}{1/p - 1/p_1}.$$

For $f \in L^p(\mathbb{R}^n)$, we define its non-increasing rearrangement f^* as in Section 6.6 of the script. Then, for $t > 0$, we let

$$f^t(x) := \begin{cases} f(x) & \text{if } |f(x)| > f^*(t^\sigma), \\ 0 & \text{otherwise} \end{cases}$$

and define $f_t := f - f^t$.

(a) Check the following properties (a drawing may help!):

- $(f^t)^*(y) \leq f^*(y)$ if $0 \leq y \leq t^\sigma$;
- $(f^t)^*(y) = 0$ if $y > t^\sigma$.

(b) Check also that

- $(f_t)^*(y) \leq f^*(t^\sigma)$ if $y \leq t^\sigma$;
- $(f_t)^*(y) \leq f^*(y)$ if $y \geq t^\sigma$.

(c) Verify that, if $f = f_1 + f_2$, then

$$(Tf)^*(t) \leq (Tf_1)^*(t/2) + (Tf_2)^*(t/2).$$

(d) Show that, if $f \in L^p(\mathbb{R}^n)$, $f^t \in L^{p_0}$ and $f_t \in L^{p_1}$.

(e) Prove the estimate

$$T(f)^*(t) \leq A_0(2/t)^{1/q_0}\|f^t\|_{L^{p_0}} + A_1(2/t)^{1/q_1}\|f_t\|_{L^{p_1}}. \tag{1}$$

(f) Using Exercise 9.1, show that

$$\|Tf\|_{L^q} \leq C\left(\int_0^\infty (t^{1/q}(Tf)^*(t))^p \frac{\mathrm{d}t}{t}\right)^{1/p} \tag{2}$$

for a constant $C > 0$.

(g) Show that, in order to prove the theorem, it is enough to show the following two estimates:

$$\left(\int_0^\infty [t^{1/q-1/q_0}\|f^t\|_{L^{p_0}}]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \leq C\|f\|_{L^p} \tag{3}$$

and

$$\left(\int_0^\infty [t^{1/q-1/q_1}\|f_t\|_{L^{p_1}}]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \leq C\|f\|_{L^p}. \tag{4}$$

(h) Show, using again Exercise 9.1, that

$$\|f^t\|_{L^{p_0}} \leq C \int_0^{t^\sigma} y^{1/p_0} f^*(y) \frac{\mathrm{d}y}{y}. \tag{5}$$

(i) Prove (3) by using (5) and the Hardy inequality (Exercise 8.4).

(j) Prove (4) by a similar argument: first find an analog of (5) and then conclude by the second Hardy inequality (Exercise 9.2).

Exercise 9.4.
Prove the following maximal function estimate for functions $f \in L \log L(\mathbb{R}^n)$: for any measurable $A \subset \mathbb{R}^n$ with finite measure,

$$\int_A |Mf|(y)\, \mathrm{d}y \leq C \int_{\mathbb{R}^n} |f|(y) \log\left(e + \mathcal{L}^n(A) \frac{|f|(y)|}{\|f\|_{L^1(\mathbb{R}^n)}}\right) \mathrm{d}y,$$

where C is a constant only depending on n. Here $L \log L$ is the space of functions $f \in L^1(\mathbb{R}^n)$ for which the right hand side is finite.

Hint: express the left hand side as an integral of the distribution function of $|Mf|$ and use inequality (5.9) from the script for large enough α (how large?).