Exercise 10.1.
Let $1 \leq p < q \leq \infty$ and $f \in L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$, where (X,μ) is a σ-finite measure space (e.g. a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure). Show that for every r such that $p < r < q$, $f \in L^r(X,\mu)$ with

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{1-\theta} \|f\|_{L^{q,\infty}}^{\theta},$$

where $\theta \in (0,1)$ is defined by

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

Exercise 10.2.
Let $f_1, \dots, f_N \in L^{p,\infty}(X,\mu)$ for $1 \leq p < \infty$. Show that

$$\left\| \sum_{j=1}^{N} f_j \right\|_{L^{p,\infty}} \leq N \sum_{j=1}^{N} \|f_j\|_{L^{p,\infty}}.$$

Exercise 10.3.
Let $0 < p < 1$. Prove that $L^p(X,\mu)$ is a complete quasi-normed space, i.e. that $\|f\|_{L^p} := \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{1/p}$ defines a quasi-norm and every quasi-norm Cauchy sequence is quasi-norm convergent.
Hint: recall the proof of the corresponding theorem for $p \geq 1$.

Exercise 10.4.
Consider the following $N!$ functions $f_\sigma : \mathbb{R} \to \mathbb{R}$, for $\sigma \in \mathcal{S}_N$, the group of permutations of $\{1, 2, \dots, N\}$:

$$f_\sigma := \sum_{j=1}^{N} \frac{N}{\sigma(j)} \chi_{\left[\frac{j-1}{N}, \frac{j}{N}\right)}.$$

(a) Show that $\|f_\sigma\|_{L^{1,\infty}} = 1$.

(b) Show that

$$\left\| \sum_{\sigma \in \mathcal{S}_n} f_\sigma \right\|_{L^{1,\infty}} = N! \left(1 + \frac{1}{2} + \dots + \frac{1}{N}\right).$$

(c) Conclude that $L^{1,\infty}(\mathbb{R})$ is not normable, i.e. that there is no norm $\|\cdot\|$ on $L^{1,\infty}(\mathbb{R})$ such that, for a constant $C \geq 1$, $C^{-1}\|f\| \leq \|f\|_{L^{1,\infty}} \leq C\|f\|$ holds for every $f \in L^{1,\infty}(\mathbb{R})$.

Exercise 10.5.
For a measurable function $g : \mathbb{R}^n \to [0,\infty)$, denote by $g^* : [0,\infty) \to [0,\infty)$ its decreasing rearrangement.
(a) Prove that for every measurable set $A \subset \mathbb{R}^n$,

$$\int_A g(x)\,\mathrm{d}x \leq \int_0^{|A|} g^*(t)\,\mathrm{d}t.$$

(b) Show the *Hardy–Littlewood inequality*: for any measurable functions $f, g : \mathbb{R}^n \to [0,\infty)$,

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \int_0^{\infty} f^*(t)g^*(t)\,\mathrm{d}t.$$