Throughout this exercise sheet we will assume that $\Omega \subset \mathbb{R}^n$ is a Lebesgue-measurable set and will work with the Lebesgue measure.

Exercise 11.1.
Given a right-continuous and non-increasing function $\varphi : (0,\infty) \to [0,\infty)$ such that $\varphi(t) = 0$ for all $t \geq |\Omega|$, prove that there exists a measurable function $f : \Omega \to [0,\infty)$ such that $f^*(t) = \varphi(t)$ for all $t > 0$.

Hint: try constructing f such that each $\{f > \lambda\}$ is of the form $B_r \cap \Omega$ for an appropriate r that guarantees that the distribution functions agree.

Exercise 11.2.
Suppose that $\varphi : (0,\infty) \to [0,\infty)$ is as in Exercise 11.1, and let $g \in L^1(\Omega)$. In this exercise we will show that there exists a function $f \in L^\infty(\Omega)$ such that $f^* = \varphi$ and equality in the Hardy–Littlewood inequality holds, namely

$$\int_\Omega fg \,\mathrm{d}x = \int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s = \int_0^\infty \varphi(s)g^*(s)\,\mathrm{d}s.$$

Denote

$$d_g(\lambda) := \big|\{|g| > \lambda\}\big| \qquad \text{and} \qquad \bar{d}_g(\lambda) := \big|\{|g| \geq \lambda\}\big|,$$

(a) Show that the set $\Lambda \subset (0,\infty)$ of those $\lambda > 0$ for which $\bar{d}_g(\lambda) - d_g(\lambda) = |\{|g| = \lambda\}| > 0$ is at most countable.

We denote $\Omega_\lambda := \{|g| = \lambda\}$ for $\lambda \in \Lambda$, and $\Omega^* := \Omega \setminus \bigcup_{\lambda\in\Lambda} \Omega_\lambda$. Then define f in the following way: for $x \in \Omega^*$, we let

$$f(x) := \operatorname{sgn}(g(x))\varphi(d_g(|g(x)|)) = \operatorname{sgn}(g(x))\varphi(\bar{d}_g(|g(x)|)).$$

For $\lambda \in \Lambda$, let $\varphi_\lambda : [0,\infty) \to [0,\infty)$ be defined by $\varphi_\lambda(t) := \varphi(t + d_g(\lambda))$ whenever $0 \leq t < |\Omega_\lambda|$ and $\varphi_\lambda(t) = 0$ for $t \geq |\Omega_\lambda|$, and apply Exercise 11.1 to obtain a measurable function $h_\lambda : \Omega_\lambda \to [0,\infty)$ such that $h_\lambda^* = \varphi_\lambda$. Then define, for $x \in \Omega_\lambda$,

$$f(x) := \operatorname{sgn}(g(x))h_\lambda(x).$$

(b) Show that $\varphi(\bar{d}_g(|g(x)|)) \leq |f(x)| \leq \varphi(d_g(|g(x)|))$ almost everywhere.

(c) Show that, for almost every $x, y \in \Omega$, $|g|(x) > |g|(y)$ implies $|f|(x) \geq |f|(y)$.

(d) Prove that $\int_\Omega fg\,\mathrm{d}x = \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t$ by examining the equality cases in Exercise 10.5.

Now it remains to show that $f^* = \varphi$. For that, it is enough to show that $d_f(\mu) = d_\varphi(\mu)$ for every $\mu > 0$. Fix one such μ and let $t > 0$ be determined by $\varphi^{-1}((\mu,\infty)) = (0,t)$.

(e) Show that $|\{x : \bar{d}_g(|g(x)|) < t\}| \leq t$. **Hint:** use the left-continuity of $\bar{d}_g$.

(f) Show that $|\{x : d_g(|g(x)|) \leq s\}| \geq s \,\forall s \geq 0$ and deduce that $|\{x : d_g(|g(x)|) < t\}| \geq t$.

(g) Show that, if the set

$$\omega := \{x : d_g(|g(x)|) < t \leq \bar{d}_g(|g(x)|)\}$$

has positive measure, then $|g|$ is equal to a constant almost everywhere in ω.

(h) Conclude by proving that $|\{|f| > \mu\}| = |\{\varphi > \mu\}| = |(0,t)| = t$.

Exercise 11.3.

The goal of this exercise is to show that the dual of $L^{p,1}(\Omega)$ for $1 < p < \infty$ is $L^{p',\infty}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. For simplicity you may assume that Ω is open.

(a) First show that any $h \in L^{p',\infty}(\Omega)$ induces a bounded linear functional

$$f \in L^{p,1}(\Omega) \mapsto \int_\Omega fh \, \mathrm{d}x \in \mathbb{R}.$$

(b) Now argue as in Exercise 8.5 and write $\Omega = \bigcup_{k\geq 1} \Omega_k$ with $\Omega_1 \Subset \Omega_2 \Subset \cdots$ and each Ω_k is open and bounded. Show that, given $T : L^{p,1}(\Omega) \to \mathbb{R}$ linear bounded, the restriction of T to $C_c(\Omega_k)$ induces a signed measure which is absolutely continuous with respect to $\mathcal{L}^n$ and therefore, by Radon–Nikodym, is represented by an L^1 function. Show moreover that these functions agree to a global function $h \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$T(\varphi) = \int_\Omega \varphi h \, \mathrm{d}x$$

whenever $\varphi \in C_c(\Omega)$.

(c) Show that $h \in L^{p',\infty}(\Omega)$ by considering integrals of the form

$$\int_\Omega \varphi h \, \mathrm{d}x$$

where $\varphi \in C_c(\Omega)$ approximates $\mathrm{sgn}(h)\mathbb{1}_{\Omega_k \cap \{|h|>\lambda\}}$ and studying the behavior of the quantities $\lambda|\{|h| > \lambda\}|^{1/p'}$ for $\lambda > 0$.

Exercise 11.4.

Let $f \in L^1(\mathbb{R}^n)$ be a function supported in the ball $B = B_1(0)$. Show that for each $k \geq 0$,

$$M(f)(\log(2 + M(f)))^k \in L^1(B) \qquad \Longleftrightarrow \qquad |f|(\log(2 + |f|))^{k+1} \in L^1(B).$$

Hint: understand the proof of Theorem 7.3 in the script and generalize it.