Exercise 12.1. ★
Let $\Omega \subset \mathbb{R}^n$ be an open set. In this exercise we will show that the dual of $L^{p,q}(\Omega)$ for $1 < p < \infty$ and $1 < q < \infty$ is $L^{p',q'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.
(a) First show that any $h \in L^{p',q'}(\Omega)$ induces a bounded linear functional

$$f \in L^{p,q}(\Omega) \mapsto \int_\Omega fh \,\mathrm{d}x \in \mathbb{R}.$$

(b) Arguing as in Exercise 11.3, given $T : L^{p,q}(\Omega) \to \mathbb{R}$ linear bounded, we obtain a function $h \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$T(\psi) = \int_\Omega \psi h \,\mathrm{d}x$$

whenever $\psi \in C_c(\Omega)$. Denote $h_{k,N} := h \mathbb{1}_{\Omega_k} \mathbb{1}_{|h| \le N}$. Show that for every $\varphi \in L^{p,q}((0,\infty))$,

$$\int_0^\infty \varphi(t) h^*_{k,N}(t) \,\mathrm{d}t \le \|T\| \, \|\varphi\|_{L^{p,q}}.$$

Hint: use Exercise 11.2.

(c) Choose now

$$\varphi(t) := \int_{t/2}^\infty s^{\frac{q'}{p'} - 1} h^*_{k,N}(s)^{q'-1} \frac{\mathrm{d}s}{s}.$$

Prove that

$$\|\varphi\|_{L^{p,q}} \le C(p,q) \|h_{k,N}\|^{q'/q}_{L^{p',q'}}.$$

(d) Show that

$$\|h^*_{k,N}\|^{q'}_{L^{p',q'}} \le C(p,q) \int_0^\infty \varphi(t) h^*_{k,N}(t) \,\mathrm{d}t.$$

(e) Conclude by showing that $h \in L^{p',q'}(\Omega)$.

Exercise 12.2. ★
Prove that there exists a function $f \in L^1(\mathbb{R}^2)$ such that, for any $u \in \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\Delta u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}^2), \tag{$\star$}$$

$\nabla^2 u \notin L^1_{\mathrm{loc}}(\mathbb{R}^2)$.
Hint: consider the function

$$f(x) := \frac{1}{|x|^2 \log^2(|x|)} \, \mathbb{1}_{B_{1/2}(0)}(x);$$

then find a *radial* solution u_0 of $(\star)$ explicitly and show that $\nabla^2 u_0 \notin L^1(B_{1/2}(0))$. Conclude for an arbitrary solution u of $(\star)$.

Exercise 12.3.
Suppose that T is the convolution operator given by a kernel K satisfying the hypotheses of Theorem 7.5. Prove that, if $f \in L^1 \log L^1(\mathbb{R}^n)$, then $Tf \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for any measurable set $A \subset \mathbb{R}^n$ with $|A| < \infty$, it holds

$$\int_A |Tf(y)| \,\mathrm{d}y \le C \int_{\mathbb{R}^n} |f(y)| \log\left(e + |A| \frac{|f(y)|}{\|f\|_{L^1(\mathbb{R}^n)}}\right) \mathrm{d}y,$$

where $C > 0$ is a constant only depending on T.
Hint: the proof of this statement uses ingredients from the proof of Theorem 7.5 (the analogous theorem but for $L^p \to L^p$) and of Theorem 5.8 (the analogous theorem but for the maximal operator instead of a convolution operator). Make sure you understand both proofs and try to combine them.

Exercise 12.4.
(a) Show that there is a constant $C \geq 1$ such that, for all $t > 0$,

$$\frac{1}{C} t(1 + \log^+(t)) \leq t \log(e + t) \leq Ct(1 + \log^+(t)).$$

(b) Show that, for a function $f : \Omega \to \mathbb{R}$, the following four properties are equivalent:

(i) $\displaystyle\int_\Omega |f| \log(e + |f|) < \infty$.

(ii) $\displaystyle\int_\Omega |f|(1 + \log^+(|f|)) < \infty$.

(iii) $\displaystyle\int_\Omega \frac{|f|}{K} \log\left(e + \frac{|f|}{K}\right) \leq 1$ for some constant $0 < K < \infty$.

(iv) $\displaystyle\int_\Omega \frac{|f|}{K} \left(1 + \log^+\left(\frac{|f|}{K}\right)\right) \leq 1$ for some constant $0 < K < \infty$.

Moreover, show that if $|\Omega| < \infty$, then the "1+" can be removed from (ii) and (iv).

(c) Let $\Phi : [0, \infty) \to [0, \infty)$ be a nonnegative, strictly increasing, continuous and convex function satisfying $\Phi(0) = 0$, and define

$$\|f\|_\Phi := \inf \left\{ 0 < K < \infty : \int_\Omega \Phi(|f|/K) \, \mathrm{d}x \leq 1 \right\},$$

where the infimum is understood to be $+\infty$ if no such K exists. Show that the set of (equivalence classes of almost everywhere equal) functions

$$L_\Phi(\Omega) := \{ f : \Omega \to \mathbb{R} \text{ measurable s.t. } \|f\|_\Phi < \infty \}$$

is a vector space and that $\| \cdot \|_\Phi$ defines a norm on $L_\Phi(\Omega)$. Show moreover that $L_\Phi(\Omega)$ is complete with respect to this norm.

(d) Deduce that the space

$$L \log L(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \int_\Omega |f| \log(e + |f|) < \infty \right\} = L^1(\Omega) \cap \left\{ f : \int_\Omega |f| \log^+ |f| < \infty \right\}$$

can be given a natural structure of Banach space.