Exercise 13.1. ★
(a) Show that for every $p \in [1, \infty]$ and $q \in [1, \infty]$, the space $L^p(\mathbb{R}^n, \ell^q)$ defines a Banach space.

(b) Show that for every $p \in (1, \infty)$ and $q \in (1, \infty)$, the dual of $L^p(\mathbb{R}^n, \ell^q)$ is $L^{p'}(\mathbb{R}^n, \ell^{q'})$

Exercise 13.2. ★
The goal of this exercise is to prove Khinchine's inequality and see an application to Fourier analysis. This states the following: let $1 \leq p < \infty$; then there exists a constant $C = C(p) > 0$ such that, for any $N \in \mathbb{N}$ and any $a_1, \ldots, a_N \in \mathbb{C}$, it holds that

$$C^{-1} \left(\sum_{j=1}^{N} |a_j|^2 \right)^{p/2} \leq \mathbb{E} \left[\left| \sum_{j=1}^{N} \epsilon_j a_j \right|^p \right] \leq C \left(\sum_{j=1}^{N} |a_j|^2 \right)^{p/2},$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the uniformly distributed random variable $(\epsilon_j)_{j=1,\ldots,N} \in \{-1, +1\}^N$. In other words, it is the average of the expression inside the $[\cdot]$ over the 2^N possible choices of signs.
(a) Prove the upper bound of Khinchine's inequality.

Hint: you may use the following fact from probability: if N, (a_j) and (ϵ_j) are as above, then for every $\lambda > 0$,

$$\mathbb{P} \left[\left| \sum_{j=1}^{N} \epsilon_j a_j \right| > \lambda \left(\sum_{j=1}^{N} |a_j|^2 \right)^{1/2} \right] \leq 4e^{-\lambda^2/2}.$$

Combine this with the usual formula for the integral using upper level sets, but interchanging measures/integrals by probabilities/expectations.

(b) Prove the lower bound of Khinchine's inequality.

Hint: bound the expectation with $p = 2$ by the expectation with a higher p and a lower p by using Hölder's inequality, and then estimate the term with the higher p by using part (a).

(c) Show that, given $p > 2$, there exist two functions $f, g : \mathbb{S}^1 \to \mathbb{C}$ such that the Fourier coefficients of f and g have the same absolute values, but such that $f \notin L^p(\mathbb{S}^1)$ and $g \in L^p(\mathbb{S}^1)$. Deduce that there is no characterization of belonging to L^p based on summability properties of the Fourier coefficients.

Exercise 13.3.
Let $1 < p < \infty$ and suppose that $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear transformation which commutes with translations. Show that there exists a function $m \in L^\infty(\mathbb{R}^n)$ such that

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi) \qquad \forall \xi \in \mathbb{R}^n$$

whenever $f \in L^2 \cap L^p$.
Hint: show this first for $p = 2$. For general p, argue by duality to conclude that T is also of type (p', p') and then apply the case $p = 2$.