Exercise 1.1.
Compute the Fourier transform of the following functions, paying attention to the spaces where they belong:
(a) $f(x) = e^{-|x|}$, $f \in L^1(\mathbb{R})$.

Solution: We compute $\hat{f}(\xi)$ directly, since the Fourier transform is defined pointwise:

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|x|} e^{-ix\xi}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty e^{-x} e^{ix\xi}\,\mathrm{d}x + \int_0^\infty e^{-x} e^{-ix\xi}\,\mathrm{d}x \right)\\ &= \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty e^{x(-1+i\xi)}\,\mathrm{d}x + \int_0^\infty e^{x(-1-i\xi)}\,\mathrm{d}x \right)\\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\xi} + \frac{1}{1+i\xi} \right) = \frac{2}{\sqrt{2\pi}\,(1+\xi^2)}.\end{aligned}$$

(b) $f(x) = \frac{\sin x}{x}$, $f \in L^2(\mathbb{R})$.

Solution: The pointwise Fourier transform does not necessarily make sense; however we know that for the sequence $f_R(x) := \chi_{[-R,R]}(x) f(x) \in L^1 \cap L^2$, $\hat{f}_R(\xi)$ makes sense pointwise and $\hat{f}_R \to \hat{f}$ in L^2; therefore if we can compute the pointwise limit $\lim_{R\to\infty} \hat{f}_R(\xi)$ for almost every $\xi \in \mathbb{R}$, we will characterize $\hat{f} \in L^2$.

The precise details of the computation belong to Complex Analysis and we will only skectch them here, since this is a course on Functional Analysis. Write

$$\begin{aligned}\hat{f}_R(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{\sin z}{z} e^{-iz\xi}\,\mathrm{d}z = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{e^{iz} - e^{-iz}}{2iz} e^{-iz\xi}\,\mathrm{d}z\\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{2\pi}} \int_{[-R,-\epsilon]\cup[\epsilon,R]} \frac{e^{i(1-\xi)z} - e^{i(-1-\xi)z}}{2iz}\,\mathrm{d}z\end{aligned}$$

and compute the integral using Cauchy's theorem and Jordan's lemma from complex analysis. More precisely, for each of the two exponentials, integrate on the upper or lower half plane depending on the sign of $1-\xi$ and $-1-\xi$, and complete the path $[-R,-\epsilon] \cup [\epsilon, R]$ by using two half circles of radii R and ϵ on the chosen half plane to enclose a bounded region. Since the functions $e^{i\alpha z}/z$ are always holomorphic away from the origin for every $\alpha \in \mathbb{C}$, there is no residue term; the term along the circle of radius R converges to zero as $R \to \infty$ thanks to Jordan's lemma, and we only get a contribution from the half circle of radius ϵ, equal to $\pm \frac{1}{\sqrt{2\pi}} \frac{\pi i}{2i}$ from each term. These terms have the same sign and therefore cancel out if $1-\xi$ and $-1-\xi$ have the same sign, which happens for $|\xi| > 1$, thus $\hat{f}(\xi) = 0$; for $|\xi| < 1$ instead they add up and we get $\pi/\sqrt{2\pi}$. In summary,

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} \chi_{[-1,1]}(\xi).$$

Alternatively one may start from this expression, compute its Fourier transform (which is straightfoward) and apply the Fourier inversion theorem to deduce that its inverse fourier transform is f.

(c) $f(x) = e^{-\frac{1}{2}|x|^2}$, $f \in \mathcal{S}(\mathbb{R}^n)$.

Solution: We first compute the Fourier transform in one dimension:

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}e^{-ix\xi}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2ix\xi)}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\xi)^2}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}\int_{-\infty+i\xi}^{\infty+i\xi} e^{-\frac{1}{2}z^2}\,\mathrm{d}z.\end{aligned}$$

This last expression just means the limit for $R\to\infty$ of the integrals from $-R+i\xi$ to $R+i\xi$ in the complex plane. Since the expression $\exp\left(-\frac{1}{2}z^2\right)$ is holomorphic in z, for each $R>0$ we may change the path of integration to $[-R,R]$; the contribution of the segments $[-R+i\xi,-R]$ and $[R,R+i\xi]$ converges exponentially to zero as $R\to\infty$. Hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,\mathrm{d}z = e^{-\frac{1}{2}\xi^2}$$

by the famous formula of Gauss. The n-dimensional version just follows from factoring the expression and applying Fubini.

Exercise 1.2. ★

In this exercise we will show the existence of a smooth partition of unity subordinate to a finite cover of a compact set.

(a) Prove that the function $g:\mathbb{R}^n\to\mathbb{R}$,

$$g(x) = \begin{cases}\exp\left(-\frac{1}{(1-|x|^2)^2}\right), & |x|<1\\ 0, & |x|\geq 1\end{cases}$$

is smooth, nonnegative, and has support in $\overline{B_1(0)}$.

Solution: Since this is a radial function and is clearly smooth around the origin, it is enough to show that it is smooth along a ray. It is also clear that $g(r)$ is smooth for all $r\neq 1$, and moreover, by iteratively computing its derivatives one sees that $g^{(k)}(r) = g(r)q_k(r)$, where $q_k(r)$ is a rational function singular at $r=1$. Since rational functions grow slower than exponentials, we get that $\lim_{r\uparrow 1} g^{(k)}(r) = 0$ and also, by induction,

$$g^{(k+1)}(1) = \lim_{r\to 1}\frac{g^{(k)}(r)}{r-1} = 0,$$

so all the derivatives of g at 1 exist and are equal to zero, hence are continuous at $r=1$ as well.

(b) Let $K\subset\mathbb{R}^n$ be a compact set, and $K\subset U\subset\mathbb{R}^n$ be an open set. Show that there exists a function $\Theta\in C_c^\infty(U)$ such that $0\leq\Theta\leq 1$ everywhere and $\Theta\equiv 1$ on K.

Hint: use convolution with a mollifier $g_\epsilon(x) := \epsilon^{-n}g(\epsilon^{-1}x)$ for $\epsilon>0$ sufficiently small.

Solution: Let $\epsilon > 0$ such that $3\epsilon < \mathrm{dist}(K, U^c)$. Introduce $\chi_\epsilon(x)$ to be the characteristic function of the open set of points which are at a distance less than ϵ from K. That is

$$\chi_\epsilon(x) := \begin{cases} 1 & \text{in case } \mathrm{dist}(x, K) < \epsilon, \\ 0 & \text{otherwise} \end{cases}.$$

Let g be as in part (a), except that we normalize it so that $\int_{\mathbb{R}^n} g = 1$. We then introduce

$$\Theta(x) := (\chi_\epsilon \star g_\epsilon)(x) = \int_{\mathbb{R}_n} \chi_\epsilon(y)\, g_\epsilon(x - y)\, \mathrm{d}y,$$

where

$$g_\epsilon(x) := \frac{1}{\epsilon^n} g\left(\frac{x}{\epsilon}\right).$$

Clearly $\Theta \in C^\infty$, since $\chi_\epsilon \in L^1(\mathbb{R}^n)$ and $g_\epsilon \in C_c^\infty(\mathbb{R}^n)$. We are now proving

i) $\Theta(x) \equiv 1$ on K.

ii) $\Theta(x) \equiv 0$ in a neighborhood of $\overline{U^c}$.

Proof of i):

$$\mathrm{supp}\ g \subset B_1(0) \Rightarrow \mathrm{supp}_x\ g_\epsilon(x) \subset B_\epsilon(0)\ ,$$

indeed

$$g_\epsilon(x) \neq 0 \Leftrightarrow \left|\frac{x}{\epsilon}\right| < 1$$

and

$$\mathrm{supp}_y\ g_\epsilon(x - y) \subset B_\epsilon(x)\ ,$$

indeed

$$g_\epsilon(x - y) \neq 0 \ \Leftrightarrow\ |x - y| < \epsilon\ .$$

Let now $x \in K$: $g_\epsilon(x - y) \neq 0$ implies $\mathrm{dist}(y, K) < \epsilon$. This gives for such an $x \in K$

$$\int_{\mathbb{R}^n} \chi_\epsilon(y)\, g_\epsilon(x - y) = \int_{\mathbb{R}^n} g_\epsilon(x - y) = \int_{\mathbb{R}^n} g(x) = 1.$$

Proof of ii): we claim that $\Theta(x) = 0$ if $\mathrm{dist}(x, U^c) < \epsilon$: if $\Theta(x) \neq 0$, it means that there is some y such that $\chi_\epsilon(y) \neq 0$ and $g_\epsilon(x - y) \neq 0$. Hence $\mathrm{dist}(y, K) < \epsilon$ and $\mathrm{dist}(x, y) < \epsilon$, which means that $\mathrm{dist}(K, U^c) \leq \mathrm{dist}(K, y) + \mathrm{dist}(y, x) + \mathrm{dist}(x, U^c) < 3\epsilon$, a contradiction.

(c) Let $K \subset \mathbb{R}^n$ be compact and $U_1, \ldots, U_p$ open subsets of $\mathbb{R}^n$ which cover K. Show that there exist functions $\Theta_1, \ldots, \Theta_p$ such that for every $i \in \{1, \ldots, p\}$, $\Theta_i \in C_c^\infty(U_i)$, $0 \leq \Theta_i \leq 1$ everywhere, and such that $\Theta_1 + \cdots + \Theta_p \equiv 1$ on K.

Solution: Let $\epsilon > 0$. From part (b), for each U_i there exists f_i with the following properties :

- $f_i \in C_c^\infty(U_i)$
- $0 \leq f_i \leq 1$
- $f_i \equiv 1$ on $\overline{(\tilde{U}_i \cap K)}$,

where

$$\tilde{U}_i := \{x \in U_i \ \mid \ \mathrm{dist}\,(x, U_i^c) > \epsilon\,\}\,.$$

We now choose $\epsilon > 0$ small enough in such a way that

$$K_\epsilon := \{x \in \mathbb{R}^n : \mathrm{dist}\ (x, K) < \epsilon\} \subset \bigcup_{i=1}^{p} \tilde{U}_i.$$

We consider $f \in C_c^\infty(K_\epsilon)$ given by part (b) such that $f \equiv 1$ on K. Finally we denote $f_0 := 1 - f$. Observe that by construction

$$\sum_{j=0}^{p} f_j \geq 1 \quad \text{ on } \mathbb{R}^n\ .$$

Let

$$\Theta_i(x) := \begin{cases} \frac{f_i(x)}{\sum_{j=0}^{p} f_j(x)} & \text{in case } x \in U_i \\ 0 & \text{otherwise} \end{cases}.$$

Then $(\Theta_i)_{i=1,\dots,p}$ satisfies the required properties.

Exercise 1.3. ★
Prove that for every $n \geq 1$ the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is separable.

Hint: approximate a function $f \in \mathcal{S}(\mathbb{R}^n)$ in each small cube by a rational number, and regularize the resulting function by mollification.

Solution: We want to produce a countable set $\mathcal{Q} \subset \mathcal{S}(\mathbb{R}^n)$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$, every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $q \in \mathcal{Q}$ such that

$$\mathcal{N}_k(f - q) = \sup_{x \in \mathbb{R}^n} \max_{j \leq k} (1 + |x|)^k |D^j(f(x) - q(x))| \leq \varepsilon$$

(this norm is equivalent to the one used in the lecture). For an integer N, let $\mathcal{P}_N$ be the set of functions which are equal to a (constant) rational number on each cube $Q_N(a) := \left[a_1, a_1 + \frac{1}{N}\right) \times \cdots \times \left[a_n, a_n + \frac{1}{N}\right)$, where a ranges over the lattice $\Lambda_N := \left(\frac{1}{N}\mathbb{Z}\right)^n$, and which vanish on all but finitely many cubes. Fix a family of smooth mollifiers $(g_\delta)_{\delta > 0}$ as in Exercise 1.2 and define

$$\mathcal{Q} := \left\{ h \star g_\delta : h \in \mathcal{P}_N, N \in \mathbb{Z}^+ \text{ and } \frac{1}{\delta} \in \mathbb{Z}^+ \right\}.$$

It is clear that $\mathcal{Q} \subset C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{Q}$ is countable. In order to show that it is dense, fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. First observe that the sequence of functions $f \star g_\delta$ converges to f in the $\mathcal{N}_k$ norm, as $\delta \to 0$: for $j \leq k$,

$$\begin{aligned} |(D^j(f \star g_\delta) - D^j f)(x)| &= |((D^j f) \star g_\delta) - D^j f)(x)| \\ &\leq \delta \sup_{B_\delta(x)} |D^{j+1} f| \leq C\delta \mathcal{N}_{k+1}(f)(1 + |x|)^{-k} \xrightarrow{\delta \downarrow 0} 0. \end{aligned}$$

Thus it is enough to show that, for a fixed small $\delta > 0$, there is a sequence $f_N \in \mathcal{P}_N$ such that $\mathcal{Q} \ni f_N \star g_\delta \to f \star g_\delta$ in the $\mathcal{N}_k$ norm as $N \to \infty$. Let $\varepsilon > 0$. For a parameter $\eta > 0$ to be determined later, let $R > 0$ be large enough so that

$$\sup_{|x|>R-1} (1+|x|)^k |f(x)| < \eta$$

(which exists since $\mathcal{N}_{k+1}(f) < +\infty$) and define f_N on $Q_N(a)$ to be some rational number at distance at most $\frac{1}{N}$ from $f(a)$ in case $Q_N(a)$ intersects B_{R-1}, and 0 in the rest of cubes. Our goal is to show that if N is large enough, then for each $0 \le j \le k$,

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^k |D^j(f_N \star g_\delta - f \star g_\delta)| = \sup_{x\in\mathbb{R}^n} (1+|x|)^k |(f_N - f) \star D^j g_\delta| \le \varepsilon. \tag{1}$$

Therefore fix one such j and let $\varphi := D^j g_\delta$ for short. It follows that

$$\sup_{|x|>R} (1+|x|)^k |(f_N - f) \star \varphi(x)| = \sup_{|x|>R} (1+|x|)^k |f \star \varphi(x)| \le C \sup_{|x|>R-1} (1+|x|)^k |f(x)| \le C\eta \le \varepsilon$$

provided that we choose η small enough. On the other hand, for $|x| \le R+1$, say $x \in Q_N(a)$. In case $Q_N(a) \cap B_{R-1} \neq \varnothing$ we have that $|f_N(x) - f(a)| \le \frac{1}{N}$, so that

$$|f(x) - f_N(x)| \le |f(x) - f(a)| + |f(a) - f_N(x)| \le \sup_{B_R} |Df| \frac{1}{N} + \frac{1}{N} \le (1+|x|)^{-k} \frac{C(R)}{N};$$

otherwise, we have that $|x| > R-1$ and $f_N(x) = 0$, therefore

$$|f(x) - f_N(x)| = |f(x)| \le (1+|x|)^{-k}\eta.$$

In any case,

$$\sup_{|x|\le R} (1+|x|)^k |(f_N - f) \star \varphi(x)| \le C(\delta) \sup_{|x|\le R+1} (1+|x|)^k |f_N(x) - f(x)| \le C(\delta) \max\left\{\eta, \frac{C(R)}{N}\right\}$$

which can be made smaller than ε by first choosing η small enough and then N large enough.

Exercise 1.4.
Show the following *baby* version of the Poincaré inequality in the Schwartz space: for every $f \in \mathcal{S}(\mathbb{R}^n)$, with $n \ge 1$, and for every $R > 0$, it holds that

$$\|f\|_{L^1(B_R(0))} \le 2R \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

Solution: For $x \in \mathbb{R}^n$, write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then, given $f \in \mathcal{S}(\mathbb{R}^n)$, thanks to the decay at infinity of Schwartz functions, we have

$$f(x', x_n) = \lim_{T\to-\infty} f(x', T) + \int_T^{x_n} \partial_{x_n} f(x', t)\,\mathrm{d}t = \int_{-\infty}^{x_n} \partial_{x_n} f(x', t)\,\mathrm{d}t,$$

and in particular

$$|f(x', x_n)| \le \int_{-\infty}^{\infty} |\nabla f(x', t)| \, \mathrm{d}t.$$

Then, by Fubini's theorem,

$$\begin{aligned} \int_{B_R^n(0)} |f(x)| \, \mathrm{d}x &\le \int_{B_R^{n-1}(0)} \int_{-R}^{R} |f(x', x_n)| \, \mathrm{d}x_n \, \mathrm{d}x' \le \int_{B_R^{n-1}(0)} \int_{-R}^{R} \int_{-\infty}^{\infty} |\nabla f(x', t)| \, \mathrm{d}t \, \mathrm{d}x_n \, \mathrm{d}x' \\ &= 2R \int_{B_R^{n-1}(0)} \int_{-\infty}^{\infty} |\nabla f(x', t)| \, \mathrm{d}t \, \mathrm{d}x' \le 2R \|\nabla f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Exercise 1.5. ♣
Show that there does not exist a function $\delta \in L^1(\mathbb{R}^n)$ (with respect to the Lebesgue measure) such that $\delta \star f = f$ for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Solution: Suppose that such a function exists. Let $g \in C_c^\infty(B_1)$ be a nonnegative function such as the one from Exercise TODO, and normalize so that $\int_{\mathbb{R}^n} g = 1$. As above, define the functions $g_\epsilon(x) := \epsilon^{-n} g(\epsilon^{-1} x)$, which are supported in B_ϵ and still have $\int_{\mathbb{R}^n} g_\epsilon = 1$ (in particular, $g_\epsilon \in \mathcal{S}(\mathbb{R}^n)$). It is well known that, for any function $h \in L^1(\mathbb{R}^n)$, $g_\epsilon \star h \to h$ almost everywhere[1]. Applying this to the function δ we see that

$$\delta(x) = \lim_{\epsilon \downarrow 0} \delta \star g_\epsilon(x) = \lim_{\epsilon \downarrow 0} g_\epsilon(x) = 0$$

for almost every $x \neq 0$, which proves that δ is zero almost everywhere, a contradiction.

[1]This holds at every Lebesgue point of h. There is an easy and convenient proof of this fact if we allow taking *a subsequence* of $\epsilon_j \downarrow 0$, which is enough for this proof: it is enough to show that $g_\epsilon \star h \to h$ in L^1, whence a subsequence converges almost everywhere. To see this, given $\eta > 0$, choose a function $\varphi \in C_c^0(\mathbb{R}^n)$ such that $\|h - \varphi\|_{L^1} < \eta$; then using the uniform continuity of φ we have

$$\begin{aligned} \limsup_{\epsilon \downarrow 0} \|h \star g_\epsilon - h\|_{L^1} &\le \limsup_{\epsilon \downarrow 0} \|(h - \varphi) \star g_\epsilon\|_{L^1} + \|\varphi \star g_\epsilon - \varphi\|_{L^1} + \|\varphi - h\|_{L^1} \\ &\le \limsup_{\epsilon \downarrow 0} \|h - \varphi\|_{L^1} \|g_\epsilon\|_{L^1} + \|\varphi \star g_\epsilon - \varphi\|_{L^1} + \|\varphi - h\|_{L^1} \\ &\le 2\eta + \limsup_{\epsilon \downarrow 0} \|\varphi \star g_\epsilon - \varphi\|_{L^1} = 2\eta, \end{aligned}$$

and we conclude since η was arbitrary.