Exercise 2.1. ★
Show that the following spaces admit natural structures of Fréchet spaces:

(a) $L^p_{\text{loc}}(\mathbb{R}^n)$.

Solution: For each $k \in \mathbb{N}$ let $\mathcal{N}_k(f) := \|f\|_{L^p(B_k)}$, where $B_k = B_k(0) \subset \mathbb{R}^n$. Then clearly $\mathcal{N}_k(f) \le \mathcal{N}_{k+1}(f)\,\forall k$ and $f \equiv 0$ (a.e.) iff $\mathcal{N}_k(f) = 0\,\forall k$. Also, if (f_j) is Cauchy with respect to each $\mathcal{N}_k$, then $f_j \to f^{(k)}$ in $L^p(B_k)$ for some $f^{(k)} \in L^p(B_k)$ thanks to the completeness of the L^p spaces. By the uniqueness of the limit, the functions $f^{(k)}$ define a global function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, and by the characterization of convergence, $f_j \to f$ in the induced distance.

(b) $C^\infty(\overline{\Omega})$, where Ω is a bounded open set in $\mathbb{R}^n$. Recall that this space is defined as the space of functions $f \in C^\infty(\Omega)$ such that each $\partial^\alpha f$ extends continuously to $\overline{\Omega}$.

Solution: For each $k \in \mathbb{N}$ let $\mathcal{N}_k(f) := \|f\|_{C^k(\overline{\Omega})}$. Then clearly $\mathcal{N}_k(f) \le \mathcal{N}_{k+1}(f)\,\forall k$ and $f \equiv 0$ iff $\mathcal{N}_0(f) = 0$. Also, if (f_j) is Cauchy with respect to each $\mathcal{N}_k$, then $f_j \to f^{(k)}$ in $C^k(\overline{\Omega})$ for some $f^{(k)} \in C^k(\overline{\Omega})$ thanks to the completeness of the space $C^k(\overline{\Omega})$. By the uniqueness of the limit, the functions $f^{(k)}$ agree and thus $f_j \to f$ in all the seminorms (and hence in the induced distance) for a function $f \in C^\infty(\overline{\Omega})$.

(c) $C^\infty(\mathbb{R}^n)$, where we do not assume any global bound on f or its derivatives.

Solution: For each $k \in \mathbb{N}$ let $\mathcal{N}_k(f) := \|f\|_{C^k(\overline{B_k})}$. Then clearly $\mathcal{N}_k(f) \le \mathcal{N}_{k+1}(f)\,\forall k$ and $f \equiv 0$ iff $\mathcal{N}_k(f) = 0\,\forall k$. Also, if (f_j) is Cauchy with respect to each $\mathcal{N}_k$, then $f_j \to f^{(k)}$ in $C^k(\overline{B_k})$ for some $f^{(k)} \in C^k(\overline{B_k})$ thanks to the completeness of the space $C^k(\overline{B_k})$. By the uniqueness of the limit, the functions $f^{(k)}$ agree and thus, for each k, $\mathcal{N}_k(f_j - f) \to 0$ (and hence $f_j \to f$ in the induced distance) for a function $f \in C^\infty(\mathbb{R}^n)$.

(d) The Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Solution: We use the norms $\mathcal{N}_k$ introduced in the lecture. Clearly $\mathcal{N}_k(f) \le \mathcal{N}_{k+1}(f)\,\forall k$ and $f \equiv 0$ iff $\mathcal{N}_0(f) = 0$. Let (f_j) be a Cauchy sequence with respect to the induced distance. Given multi-indices α and β, it follows that the functions $x^\alpha \partial^\beta f_j$ are a Cauchy sequence in $C_b(\mathbb{R}^n) = L^\infty \cap C^0$ and hence converge uniformly to some function $f_{\alpha,\beta}$. Letting $f := f_{0,0} \in C_b(\mathbb{R}^n)$, by uniqueness of the limit, we have the compatibility conditions that $f_{\alpha,\beta} = x^\alpha \partial^\beta f$. Recalling the definition of $\mathcal{N}_k$ we see that in fact $\mathcal{N}_k(f_j - f) \to 0$ for each k, which proves the completeness.

Exercise 2.2. ★
Show that $C^\infty_c(\mathbb{R}^n)$ does **not** admit the structure of a Fréchet space. More precisely, prove the following statement: suppose that $(\mathcal{N}_p)_{p\in\mathbb{N}}$ is a family of seminorms on $C^\infty_c(\mathbb{R}^n)$ and d denotes the induced distance. Assume that for every sequence $(\varphi_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)$ converging to some $\varphi \in C^\infty_c(\mathbb{R}^n)$ with respect to d, also $\varphi_j \to \varphi$ *pointwise*[1]. Then show that $(C^\infty_c(\mathbb{R}^n), d)$ cannot be complete.

[1]One has to assume some kind of compatibility between the norms $\mathcal{N}_p$ and the usual convergence of functions, otherwise $C^\infty_c(\mathbb{R}^n)$ is just an abstract vector space and one can put even a Banach structure on it. In fact, it is enough to just require that $\varphi_j \rightharpoonup \varphi$ *in the sense of distributions.*

Solution: Fix a sequence of nonnegative functions $\varphi_k \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi_k > 0$ on B_k, and choose a sequence of positive real numbers a_k such that

$$\mathcal{N}_p(a_k\varphi_k) \leq 2^{-k} \qquad \text{for } 1 \leq p \leq k.$$

Then the sequence $\psi_k := \sum_{j\leq k} a_j\varphi_j$ is Cauchy with respect to all the $\mathcal{N}_p$ seminorms: for every p, if we take $m \geq n \geq p$ then

$$\mathcal{N}_p(a_n\varphi_n + \cdots + a_m\varphi_m) \leq \mathcal{N}_p(a_n\varphi_n) + \cdots + \mathcal{N}_p(a_m\varphi_m) \leq 2^{-n} + \cdots + 2^{-m} \leq 2^{-n+1} \xrightarrow{n\to\infty} 0.$$

Hence (ψ_k) is Cauchy with respect to d, and if we assume that $(C_c^\infty(\mathbb{R}^n), d)$ is complete, it converges to some function $\psi \in C_c^\infty(\mathbb{R}^n)$ with respect to the distance d. It follows from the compatibility assumption that $\psi_k \to \psi$ also pointwise. Pick $R > 0$ such that $\operatorname{supp}\psi \subset B_R$ and a point $x \in \mathbb{R}^n \setminus B_R$. Then, by definition, $\psi_k(x)$ is an increasing sequence of positive numbers and cannot converge to $\psi(x) = 0$.

Exercise 2.3.

(a) Show that the function $f(x) = e^x$ is not a tempered distribution, i.e. that there exists no $T \in \mathcal{S}'(\mathbb{R})$ such that $\langle T, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$ for every $\varphi \in C_c^\infty(\mathbb{R})$.

Solution: Suppose that such $T \in \mathcal{S}'(\mathbb{R})$ exists and is of order m. Fix $\varphi \in C_c^\infty(\mathbb{R})$ and let $\varphi_a(y) := \varphi(y - a)$ for $a > 0$. Then

$$\mathcal{N}_m(\varphi_a) = \sup_{y\in\mathbb{R}} \max_{0\leq i,j\leq m} |y|^i|\partial^j\varphi_a(y)| = \sup_{x\in\operatorname{supp}\varphi} \max_{0\leq i,j\leq m} |a + x|^i|\partial^j\varphi(x)| \leq C(1 + a)^m,$$

where C depends on φ and m. Then

$$\int_{\mathbb{R}} e^y\varphi_a(y)\,\mathrm{d}y = \int_{\mathbb{R}} e^{x+a}\varphi(x)\,\mathrm{d}x = e^a \int_{\mathbb{R}} e^x\varphi(x)\,\mathrm{d}x = \lambda e^a,$$

where $\lambda > 0$ if we choose φ nonnegative. This clearly cannot be bounded by a constant times $\mathcal{N}_m(\varphi_a)$ for $a > 0$ large enough.

(b) Show that the function $g(x) = e^x\cos(e^x)$ does define a tempered distribution (in the above sense).

Solution: It is clear that $g(x)$ is the derivative of $h(x) = \sin(e^x)$, which is in L^∞ and hence defines a tempered distribution. Since $h \in C^1(\mathbb{R})$ and the distributional derivative agrees with the classical derivative for $C^1 \cap \mathcal{S}'$ functions, $g = h' \in \mathcal{S}'$.

Exercise 2.4.

Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi \equiv 0$ on $\operatorname{supp} T$. Is it true that $\langle T, \varphi\rangle = 0$?

Solution: No! Just take $T = -\delta'$ and $\varphi(t) = t\chi(t)$, where $\chi \in C_c^\infty((-2,2))$ is a cutoff function with $\chi \equiv 1$ on $[-1,1]$. Then $\operatorname{supp} T = \{0\}$ and $\varphi(0) = 0$, but $\langle T, \varphi\rangle = \varphi'(0) = 1$.

Exercise 2.5.
For $\lambda > 0$, let $T_\lambda : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\langle T_\lambda, \varphi\rangle := \lim_{\varepsilon\to 0} \int_{(-\infty,-\varepsilon]\cup[\lambda\varepsilon,\infty)} \frac{\varphi(t)}{t}\,\mathrm{d}t.$$

Show that T_λ is a tempered distribution and relate it to $T_1 = \mathrm{p.\,v.}\,\frac{1}{t}$.

Solution: Suppose that $\lambda > 1$ (otherwise the proof is analogous). Then

$$\langle T_\lambda, \varphi\rangle = \lim_{\varepsilon\to 0} \int_{(-\infty,-\varepsilon]\cup[\lambda\varepsilon,\infty)} \frac{\varphi(t)}{t}\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_{(-\infty,-\varepsilon]\cup[\varepsilon,\infty)} \frac{\varphi(t)}{t}\,\mathrm{d}t - \int_{[\varepsilon,\lambda\varepsilon]} \frac{\varphi(t)}{t}\,\mathrm{d}t.$$

The first integral converges to $\langle \mathrm{p.\,v.}\,\frac{1}{t}, \varphi\rangle$ and for the second one we have

$$\int_{[\varepsilon,\lambda\varepsilon]} \frac{\varphi(t)}{t}\,\mathrm{d}t = \int_{[\varepsilon,\lambda\varepsilon]} \frac{\varphi(t)-\varphi(0)}{t}\,\mathrm{d}t + \varphi(0)\int_\varepsilon^{\lambda\varepsilon} \frac{\mathrm{d}t}{t} = \int_{[\varepsilon,\lambda\varepsilon]} \frac{\varphi(t)-\varphi(0)}{t}\,\mathrm{d}t + \varphi(0)\log\lambda.$$

Finally, since

$$\left|\int_{[\varepsilon,\lambda\varepsilon]} \frac{\varphi(t)-\varphi(0)}{t}\,\mathrm{d}t\right| \le \int_{[\varepsilon,\lambda\varepsilon]} \left|\frac{\varphi(t)-\varphi(0)}{t}\right|\,\mathrm{d}t \le \lambda\varepsilon\|\varphi'\|_{L^\infty} \xrightarrow{\varepsilon\to 0} 0,$$

we obtain that $T_\lambda = \mathrm{p.\,v.}\,\frac{1}{t} - \log\lambda\,\delta_0$ and hence $T_\lambda \in \mathcal{S}'(\mathbb{R})$.

Exercise 2.6. ♣
Let $1 < p < \infty$ and $(u_j)_{j\in\mathbb{N}} \subset L^p(\mathbb{R})$ be a sequence of functions such that $u_j \rightharpoonup u$ in $\mathcal{S}'(\mathbb{R})$ for a function $u \in L^p(\mathbb{R})$.

(a) Show that

$$u_j \rightharpoonup u \text{ in } L^p(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad \|u_j\|_{L^p} \le C \text{ for some constant } C > 0.$$

Solution: The "$\Rightarrow$" part follows from the Banach–Steinhaus theorem: if q denotes the conjugate exponent of p, for each $f \in L^q(\mathbb{R})$ we have that $\langle u_j, f\rangle$ is converging, hence bounded, and therefore u_j are uniformly bounded as linear functionals $L^q \to \mathbb{R}$, which gives that

$$\|u_j\|_{L^p} = \|u_j\|_{(L^q)^*} \le C.$$

For the "$\Leftarrow$" diretion, observe first that by the lower semicontinuity of the norms, $\|u\|_{L^p} \leq C$ too. Indeed we have that

$$\|u\|_{L^p} = \|u\|_{(L^q)^*} = \sup_{\varphi \in \mathcal{S}(\mathbb{R})} \frac{\langle u, \varphi \rangle}{\|\varphi\|_{L^q}} = \sup_{\varphi \in \mathcal{S}(\mathbb{R})} \limsup_{j \to \infty} \frac{\langle u_j, \varphi \rangle}{\|\varphi\|_{L^q}} \leq C$$

thanks to the density of $\mathcal{S}$ in L^q.

Now fix $f \in L^q$, let $\varepsilon > 0$ and choose $\varphi \in \mathcal{S}(\mathbb{R})$ be such that $\|f - \varphi\|_{L^q} \leq \varepsilon/3C$ (which exists by the density of Schwartz functions). Then let j_0 be large enough so that $|\langle u_j, \varphi \rangle - \langle u, \varphi \rangle| \leq \varepsilon/3$ for all $j \geq j_0$. Finally, for all such j we have

$$\begin{aligned} |\langle u_j, f \rangle - \langle u, f \rangle| &\leq |\langle u_j, f - \varphi \rangle| + |\langle u_j, \varphi \rangle - \langle u, \varphi \rangle| + |\langle u, \varphi - f \rangle| \\ &\leq C\|f - \varphi\|_{L^q} + \frac{\varepsilon}{3} + C\|\varphi - f\|_{L^q} \leq \varepsilon. \end{aligned}$$

(b) For each $1 < p < \infty$, construct one such sequence that does not satisfy the two equivalent conditions from part (a), that is, $u_j, u \in L^p(\mathbb{R})$ such that $u_j \rightharpoonup u$ in $\mathcal{S}'$ but not in L^p.

Solution: There are many possible constructions, one is as follows: let $u_j = j(\chi_{[0,j^{-1}]} - \chi_{[-j^{-1},0]})$, so that

$$\|u_j\|_{L^p} = j \left(\frac{2}{j} \right)^{1/p} = 2^{1/p} j^{1-1/p} \xrightarrow{j \to \infty} \infty$$

for every $1 < p < \infty$. However, given any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u_j, \varphi \rangle = j \left(\int_0^{1/j} \varphi(t)\,\mathrm{d}t - \int_{-1/j}^0 \varphi(t)\,\mathrm{d}t \right) = \int_0^1 \varphi\left(\frac{x}{j}\right)\,\mathrm{d}x - \int_{-1}^0 \varphi\left(\frac{x}{j}\right)\,\mathrm{d}x \xrightarrow{j \to \infty} \varphi(0) - \varphi(0) = 0.$$

This shows that $u_j \rightharpoonup 0$ in $\mathcal{S}'$, but by part (a) not in L^p since $\|u_j\|_{L^p}$ is unbounded.

Exercise 2.7.

Consider the evaluation map $\mathrm{ev}_0 : C_b^0(\mathbb{R}) \to \mathbb{R}$, $\mathrm{ev}_0(\varphi) = \varphi(0)$, where $C_b^0(\mathbb{R})$ denotes the Banach space of all continuous and bounded functions on $\mathbb{R}$.

(a) Show that there exists a linear bounded extension $T : L^\infty(\mathbb{R}) \to \mathbb{R}$ of ev_0.

Solution: This is an immediate consequence of Hahn–Banach.

(b) Show that any such $T \in L^\infty(\mathbb{R})^*$ does not correspond to any function in $L^1(\mathbb{R})$. As a result, the embedding $L^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})^{**} = L^\infty(\mathbb{R})^*$ is far from being surjective.

Solution: We simply show that there is no function $f \in L^1(\mathbb{R})$ such that $\int f\varphi = \varphi(0)$ for every $\varphi \in C_c(\mathbb{R})$. Indeed, by a standard argument using bump functions, f should vanish almost everywhere on any open set U not containing 0, thus $f \equiv 0$ almost everywhere, which is a contradiction.