Exercise 3.1. ★
Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and let $p \in \mathbb{N}$ be the order of T. Consider moreover $\varphi \in C^\infty(\mathbb{R}^n)$, such that $\partial^\alpha \varphi = 0$ on $\operatorname{supp} T$ for any α such that $|\alpha| \leq p$. Show that

$$\langle T, \varphi \rangle = 0.$$

Hint: denoting $K := \operatorname{supp} T$ and $K_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \delta\}$, define for every small enough $\varepsilon > 0$ the function $\psi_\varepsilon := \mathbb{1}_{K_{2\varepsilon}} \star \chi_\varepsilon$, where $(\chi_\varepsilon)_{\varepsilon>0}$ is a standard family of mollifiers, and given any $\varphi \in C^\infty(\mathbb{R}^n)$, investigate the behavior of $\langle T, \varphi\psi_\varepsilon \rangle$ as $\varepsilon \to 0$.

Solution:

Let $K := \operatorname{supp} T$, which is compact, and define $\mathbb{1}_{K_{2\varepsilon}}, \chi_\varepsilon$ and ψ_ε as in the hint. Here we are assuming that

$$\chi \in C_c^\infty(B_1(0)), \quad \int_{\mathbb{R}^n} \chi = 1 \quad \text{and} \quad \chi_\varepsilon(s) = \frac{1}{\varepsilon^n} \chi\left(\frac{s}{\varepsilon}\right).$$

We have that $\psi_\varepsilon \in C^\infty(\mathbb{R}^n)$ and from the definition of the convolution operator we have

$$\operatorname{supp}\left(\mathbb{1}_{K_{2\varepsilon}} \star \chi_\varepsilon\right) \subset \operatorname{supp}\left(\mathbb{1}_{K_{2\varepsilon}}\right) + \operatorname{supp}\left(\chi_\varepsilon\right) \subset \operatorname{supp}\left(\mathbb{1}_{K_{3\varepsilon}}\right) .$$

Hence we deduce

$$\psi_\varepsilon \in C_0^\infty(\mathbb{R}^n) .$$

We claim that $\psi_\varepsilon \equiv 1$ on K_ε. Indeed, let $x \in K_\varepsilon$, there holds

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^n} \mathbb{1}_{K_{2\varepsilon}}(y) \frac{1}{\varepsilon^n} \chi\left(\frac{x-y}{\varepsilon}\right) dy$$

Observe that $\operatorname{supp}\left(\chi\left(\frac{\cdot}{\varepsilon}\right)\right) \subset B_\varepsilon(0)$. Hence, for $\chi\left(\frac{x-y}{\varepsilon}\right) \neq 0$ there need to be $|x-y| < \varepsilon$ which implies $\operatorname{dist}(y, K_\varepsilon) < \varepsilon$ and hence $y \in K_{2\varepsilon}$. Therefore

$$\psi_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \chi\left(\frac{x-y}{\varepsilon}\right) dy = 1$$

which concludes the proof of the claim.

We decompose φ as follows

$$\varphi = \varphi\, \psi_\varepsilon + (1 - \psi_\varepsilon)\varphi.$$

From the claim we just proved we deduce $\operatorname{supp}(1 - \psi_\varepsilon)\varphi \subset K_\varepsilon^c$. Hence $\langle T,\ (1 - \psi_\varepsilon)\, \varphi \rangle = 0$ since $\operatorname{supp} T = K$. Thus we have $\langle T,\ \varphi \rangle = \langle T,\ \varphi\psi_\varepsilon \rangle$.

We claim that for any $\alpha \in \mathbb{N}^n$ there exists of $C_\alpha > 0$ such that

$$\|\partial^\alpha \psi_\varepsilon\|_\infty \leq C_\alpha \varepsilon^{-|\alpha|} :$$

Indeed, we have on one hand

$$\partial^\alpha (\mathbb{1}_{K_{2\varepsilon}} \star \chi_\varepsilon) = \mathbb{1}_{K_{2\varepsilon}} \star \partial^\alpha \chi_\varepsilon ,$$

and on the other hand a direct computation gives

$$\partial^\alpha \chi_\varepsilon = \varepsilon^{-n-|\alpha|}(\partial^\alpha \chi) \Longrightarrow \|\partial^\alpha \chi_\varepsilon\|_1 = \varepsilon^{-|\alpha|} C_\alpha .$$

Combining these two facts we obtain

$$\begin{aligned}|\partial^\alpha \psi_\varepsilon(x)|_\infty &= \left\| \int_{\mathbb{R}^n} \mathbb{1}_{K_{2\varepsilon}}(y) \, \frac{1}{\varepsilon^n} \, \partial_x^\alpha \chi \left(\frac{x-y}{\varepsilon} \right) \, dy \right\|_\infty \\ &\leq \|\mathbb{1}_{K_{2\varepsilon}}\|_\infty \, \|\partial^\alpha \chi_\varepsilon\|_1 \leq \varepsilon^{-|\alpha|} C_\alpha \, .\end{aligned}$$

This implies the claim.

Let $x \in K_{3\varepsilon}$, we consider $y \in K$ such that $|x-y| \leq 4\varepsilon$. Taylor expansion at y gives for any γ with $|\gamma| \leq p$ the existence of ξ in the segment $[x, y]$ such that

$$\partial^\gamma \varphi(x) = \partial^\gamma \varphi(y) + \sum_{\substack{|\alpha| \leq p \\ \gamma < \alpha}} \partial^\alpha \varphi(y) \, \frac{h^{\alpha-\gamma}}{(\alpha-\gamma)!} + \sum_{|\beta| = p+1} \partial^\beta \varphi(\xi) \, \frac{h^{\beta-\gamma}}{(\beta-\gamma)!}$$

where

$$\frac{1}{\alpha!} = \frac{1}{\alpha_1! \ldots \alpha_n!}$$

and

$$x - y = (h_1, \ldots, h_n), \; h^\alpha = h_1^{\alpha_1} \cdots h_n^{\alpha_n} \, .$$

From the hypothesis we have for any $y \in K = \operatorname{supp}(T)$ $\partial^\gamma \varphi(y) = 0 \quad \forall |\gamma| \leq p$. Combining this hypothesis with the Taylor expansion we obtain

$$\|\partial^\gamma \varphi\|_{L^\infty(K_{3\varepsilon})} \leq C_\varphi \; \varepsilon^{p+1-|\gamma|} \, .$$

Finally we bound

$$\begin{aligned}|\langle T, \; \varphi \rangle| = |\langle T, \; \varphi \, \psi_\varepsilon \rangle| &\leq C \sum_{|\alpha| \leq p} \|\partial^\alpha (\varphi \, \psi_\varepsilon)\|_{L^\infty(K_{3\varepsilon})} \\ &\leq C \cdot \sum_{|\alpha| \leq p} \sum_{|\gamma| \leq |\alpha|} \|\partial^\gamma \varphi\|_{L^\infty(K_{3\varepsilon})} \|\partial^{\alpha-\gamma} \, \psi_\varepsilon\|_\infty C_\gamma \\ &\leq C \sum_{|\alpha| \leq p} C'' \varepsilon^{p+1-|\alpha|} \leq C_{\varphi, T} \, \varepsilon \, .\end{aligned}$$

This holds for any arbitrary small ε hence we deduce $|\langle T, \; \varphi \rangle| = 0$. □

Exercise 3.2. ★

We have seen in the lecture that for every $T \in \mathcal{S}'(\mathbb{R}^n)$, there is a well-defined map

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto T \star \varphi \in C^\infty(\mathbb{R}^n).$$

(a) Show that this map is continuous as a linear map between Fréchet spaces.

Solution: Suppose that the order of T is p and let $k \in \mathbb{N}$. Then

$$\begin{aligned}\sup_{\substack{x\in B_k\\|\alpha|\le k}} |\partial^\alpha (T\star\varphi)(x)| &= \sup_{\substack{x\in B_k\\|\alpha|\le k}} |(T\star\partial^\alpha\varphi)(x)| = \sup_{\substack{x\in B_k\\|\alpha|\le k}} |\langle T, (\partial^\alpha\varphi)(x-\cdot)\rangle| \\ &\le \sup_{\substack{x\in B_k\\|\alpha|\le k}} C\mathcal{N}_p((\partial^\alpha\varphi)(x-\cdot)) \\ &\le C \sup_{\substack{x\in B_k\\|\alpha|\le k}} \sup_{\substack{y\in\mathbb{R}^n\\|\beta|\le p}} (1+|y|)^p |\partial^{\alpha+\beta}\varphi(x-y)|.\end{aligned}$$

Now observe that for $x \in B_k$ and $y \in \mathbb{R}^n$, $1+|y| \le 1+|x|+|x-y| \le k+1+|x-y| \le (k+1)(1+|x-y|)$, hence

$$\sup_{\substack{x\in B_k\\|\alpha|\le k}} |\partial^\alpha(T\star\varphi)(x)| \le C \sup_{\substack{z\in\mathbb{R}^n\\|\gamma|\le k+p}} (1+|z|)^p|\partial^\gamma\varphi(z)| \le C\mathcal{N}_{k+p}(\varphi).$$

(b) Show that, if in addition $T \in \mathcal{E}'(\mathbb{R}^n)$, then $T\star\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(c) In this case, prove that moreover the map is continuous into the space $\mathcal{S}(\mathbb{R}^n)$.

Solution: Suppose that T is supported in B_R and has order p. Then for each $k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{N}_k(T\star\varphi) &= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le k}} (1+|x|)^k|\partial^\alpha(T\star\varphi)(x)| = \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le k}} (1+|x|)^k|(T\star\partial^\alpha\varphi)(x)| \\ &= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le k}} (1+|x|)^k|\langle T, (\partial^\alpha\varphi)(x-\cdot)\rangle| \\ &\le \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le k}} (1+|x|)^k C \sup_{\substack{y\in B_R\\|\beta|\le p}} |(\partial^{\alpha+\beta}\varphi)(x-y)|.\end{aligned}$$

Now observe that $1+|x| \le 1+|x-y|+|y| \le 1+r+|x-y| \le (1+r)(1+|x-y|)$ if $y \in B_R$, so

$$\mathcal{N}_k(T\star\varphi) \le C \sup_{\substack{z\in\mathbb{R}^n\\|\gamma|\le k+p}} (1+|z|)^k|\partial^\gamma\varphi(z)| \le C\mathcal{N}_{k+p}(\varphi).$$

Exercise 3.3.

Recall the definition of the translation operators: for $a \in \mathbb{R}^n$ we first define $\tau_a\varphi(x) = \varphi(x-a)$ for functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and then we define by duality $\langle \tau_a T, \varphi\rangle := \langle T, \tau_{-a}\varphi\rangle$ for $T \in \mathcal{S}'(\mathbb{R}^n)$.

(a) ★ Prove that $\forall T \in \mathcal{S}'(\mathbb{R}^n), \forall\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\forall a \in \mathbb{R}^n$ it holds:

$$\tau_a(T\star\varphi) = (\tau_a T)\star\varphi = T\star\tau_a\varphi.$$

Solution: One one hand:

$$\begin{aligned}\tau_a(T\star\varphi)(x) &= (T\star\varphi)(x-a) = \langle T,\varphi(x-a-\cdot)\rangle = \langle T,\varphi(x-(a+\cdot))\rangle\\ &= \langle T,\tau_{-a}(\varphi(x-\cdot))\rangle = \langle \tau_a T,\varphi(x-\cdot)\rangle = ((\tau_a T)\star\varphi)(x).\end{aligned}$$

On the other hand:

$$\tau_a(T\star\varphi)(x) = (T\star\varphi)(x-a) = \langle T,\varphi(x-a-\cdot)\rangle = \langle T,(\tau_a\varphi)(x-\cdot)\rangle = (T\star(\tau_a\varphi))(x).$$

(b) Let $U:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ be a linear continuous map commuting with translations, that is, such that for any $a\in\mathbb{R}^n$, $U\circ\tau_a=\tau_a\circ U$. Prove that there exists a $T\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$U\varphi = T\star\varphi \qquad \forall\varphi\in\mathcal{S}(\mathbb{R}^n).$$

Solution: Let $\langle T,\varphi\rangle := (U\check{\varphi})(0)$ for $\varphi\in\mathcal{S}(\mathbb{R}^n)$ (here $\check{\varphi}(x)=\varphi(-x)$). Notice that, owing to the continuity of U, $|(U\check{\varphi})(0)|\le\mathcal{N}_0(U\check{\varphi})\le C\mathcal{N}_p(\check{\varphi})=C\mathcal{N}_p(\varphi)$, so T is indeed a tempered distribution. We just need to check:

$$U\varphi(a) = (\tau_{-a}(U\varphi))(0) = (U\tau_{-a}\varphi)(0) = \langle T,\widetilde{\tau_{-a}\varphi}\rangle = \langle T,\varphi(a-\cdot)\rangle = (T\star\varphi)(a),$$

where we have used that $\widetilde{\tau_{-a}\varphi}(x)=(\tau_{-a}\varphi)(-x)=\varphi(-x+a)=\varphi(a-x)$.

Exercise 3.4.

(a) Determine all the tempered distributions $T\in\mathcal{S}'(\mathbb{R})$ such that $tT=1$ (here t is the independent variable of $\mathbb{R}$ and 1 denotes the constant function 1, seen as a distribution).

Solution: Let $P=\mathrm{p.\,v.}\,\frac{1}{t}\in\mathcal{S}'(\mathbb{R})$; in the lecture we have seen that $tP=1$, therefore for any such T it holds that $t(P-T)=0$. Given any $\varphi\in C_c^\infty(\mathbb{R}\setminus\{0\})$, also $\varphi(t)/t$ is in $C_c^\infty(\mathbb{R}\setminus\{0\})$, hence

$$\langle P-T,\varphi\rangle = \left\langle P-T,t\frac{\varphi}{t}\right\rangle = \left\langle t(P-T),\frac{\varphi}{t}\right\rangle = 0,$$

which shows that $\mathrm{supp}(P-T)\subseteq\{0\}$. Let m be the order of $P-T$; then by the lemma of Schwartz we have that $P-T=\sum_{j=0}^m c_j\delta_0^{(j)}$ for some $m\in\mathbb{N}$ and coefficients $c_j\in\mathbb{R}$.
Now take arbitrary coefficients $a_1,\dots,a_m$ and consider the polynomial $q(t)=\sum_{k=1}^m a_kt^k$, which is in $C^\infty(\mathbb{R})$. Since $P-T\in\mathcal{E}'(\mathbb{R})$, we may pair them and get

$$\begin{aligned}0 &= \left\langle t(P-T),\sum_{k=1}^m a_kt^{k-1}\right\rangle = \left\langle P-T,\sum_{k=1}^m a_kt^k\right\rangle = \left\langle \sum_{j=0}^m c_j\delta_0^{(j)},\sum_{k=1}^m a_kt^k\right\rangle\\ &= \sum_{j=0}^m\sum_{k=1}^m c_ja_k(-1)^j\left(\frac{d}{dt}\right)^j t^k\Big|_{t=0} = \sum_{j=0}^m\sum_{k=1}^m c_ja_k(-1)^jk!\delta_{jk} = \sum_{k=1}^m c_ka_k(-1)^kk!\end{aligned}$$

Choosing arbitrary values for $a_1,\dots,a_m$ we see that $c_1=\dots=c_m=0$, thus $P-T=c_0\delta_0$ and in fact any c_0 works, since $\langle t\delta_0,\varphi\rangle=\langle\delta_0,t\varphi\rangle=0$. Thus

$$\{T\in\mathcal{S}'(\mathbb{R}):tT=1\} = \left\{\mathrm{p.\,v.}\,\frac{1}{t}+c\delta_0 : c\in\mathbb{R}\right\}.$$

(b) Does there exist any tempered distribution $S \in \mathcal{S}'(\mathbb{R})$ such that $t^2 S = 1$?

Solution: Yes! Let $P = \mathrm{p.\,v.}\,\frac{1}{t}$, multiply by t the equation $tP = 1$ and differentiate:

$$t^2 P = t \quad\Longrightarrow\quad 2tP + t^2 P' = 1 \quad\Longrightarrow\quad 2 + t^2 P' = 1 \quad\Longrightarrow\quad t^2(-P') = 1.$$

Here we have used the Leibniz rule for the product of a smooth function and a distribution, which is immediate to prove. Thus $S = -P'$ is one such distribution. One can also give an explicit expression for $T = -P'$:

$$\begin{aligned}\langle T, \varphi\rangle = \langle P, \varphi'\rangle &= \lim_{\varepsilon\to 0}\int_{|t|>\varepsilon}\frac{\varphi'(t)}{t}\,\mathrm{d}t = \lim_{\varepsilon\to 0}\frac{-\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon} + \int_{|t|>\varepsilon}\frac{\varphi(t)}{t^2}\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\frac{2\varphi(0)-\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon} - \frac{2\varphi(0)}{\varepsilon} + \int_{|t|>\varepsilon}\frac{\varphi(t)}{t^2}\,\mathrm{d}t \\ &= -\varphi'(0) + \varphi'(0) + \lim_{\varepsilon\to 0} -\frac{2\varphi(0)}{\varepsilon} + \int_{|t|>\varepsilon}\frac{\varphi(t)}{t^2}\,\mathrm{d}t = \lim_{\varepsilon\to 0} -\frac{2\varphi(0)}{\varepsilon} + \int_{|t|>\varepsilon}\frac{\varphi(t)}{t^2}\,\mathrm{d}t.\end{aligned}$$

Exercise 3.5.

(a) Given a rotation $A \in \mathsf{SO}(n)$, define by duality the rotation operator $\mathsf{R}_A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ for tempered distributions, extending the rotation operator $\mathsf{R}_A f(x) := f(Ax)$ of functions. How is the Fourier transform of $\mathsf{R}_A T$ related to $\widehat{T}$?

Solution: We let $\langle \mathsf{R}_A T, \varphi\rangle := \langle T, \mathsf{R}_{A^{-1}}\varphi\rangle$. Then, if T is given by an L^1_{loc} function f,

$$\langle \mathsf{R}_A f, \varphi\rangle = \langle f, \mathsf{R}_{A^{-1}}\varphi\rangle = \int f(x)\varphi(A^{-1}x)\,\mathrm{d}x = \int f(Ay)\varphi(y)\,\mathrm{d}y$$

agrees with $\langle \mathsf{R}_A f, \varphi\rangle$ defined in the pointwise way. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have that

$$\begin{aligned}\widehat{\mathsf{R}_A\varphi}(\xi) &= \int \mathsf{R}_A\varphi(y)e^{-iy\cdot\xi}\,\mathrm{d}y = \int \varphi(Ay)e^{-iAy\cdot A\xi}\,\mathrm{d}y \\ &= \int \varphi(x)e^{-ix\cdot A\xi}\,\mathrm{d}x = \hat{\varphi}(A\xi) = \mathsf{R}_A\hat{\varphi}(\xi),\end{aligned}$$

so for $T \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\langle \widehat{\mathsf{R}_A T}, \varphi\rangle = \langle \mathsf{R}_A T, \widehat{\varphi}\rangle = \langle T, \mathsf{R}_{A^{-1}}\widehat{\varphi}\rangle = \langle T, \widehat{\mathsf{R}_{A^{-1}}\varphi}\rangle = \langle \widehat{T}, \mathsf{R}_{A^{-1}}\varphi\rangle = \langle \mathsf{R}_A\widehat{T}, \varphi\rangle,$$

so $\widehat{\mathsf{R}_A T} = \mathsf{R}_A\widehat{T}$.

(b) Given a scalar $\lambda > 0$, define by duality the dilation operator $\mathsf{D}_\lambda : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ for tempered distributions, extending the dilation operator $\mathsf{D}_\lambda f(x) := f(\lambda x)$ of functions. How is the Fourier transform of $\mathsf{D}_\lambda T$ related to $\widehat{T}$?

Solution: We let $\langle \mathsf{D}_\lambda T, \varphi\rangle := \lambda^{-n}\langle T, \mathsf{D}_{\lambda^{-1}}\varphi\rangle$. Then, if T is given by an L^1_{loc} function f,

$$\langle \mathsf{D}_\lambda f, \varphi\rangle = \langle f, \mathsf{D}_{\lambda^{-1}}\varphi\rangle = \lambda^{-n}\int f(x)\varphi(\lambda^{-1}x)\,\mathrm{d}x = \int f(\lambda y)\varphi(y)\,\mathrm{d}y$$

agrees with $\langle \mathsf{D}_\lambda f, \varphi\rangle$ defined in the pointwise way. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have that

$$\begin{aligned}\widehat{\mathsf{D}_\lambda\varphi}(\xi) &= \int \mathsf{D}_\lambda\varphi(y)e^{-iy\cdot\xi}\,\mathrm{d}y = \int \varphi(\lambda y)e^{-i\lambda y\cdot\lambda^{-1}\xi}\,\mathrm{d}y \\ &= \lambda^{-n}\int \varphi(x)e^{-ix\cdot\lambda^{-1}\xi}\,\mathrm{d}x = \lambda^{-n}\hat{\varphi}(\lambda^{-1}\xi) = \lambda^{-n}\mathsf{D}_{\lambda^{-1}}\hat{\varphi}(\xi),\end{aligned}$$

so for $T \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\langle \widehat{\mathsf{D}_\lambda T}, \varphi\rangle = \langle \mathsf{D}_\lambda T, \widehat{\varphi}\rangle = \lambda^{-n}\langle T, \mathsf{D}_{\lambda^{-1}}\widehat{\varphi}\rangle = \langle T, \widehat{\mathsf{D}_\lambda\varphi}\rangle = \langle \widehat{T}, \mathsf{D}_\lambda\varphi\rangle = \lambda^{-n}\langle \widehat{T}, \lambda^n\mathsf{D}_\lambda\varphi\rangle = \lambda^{-n}\langle \mathsf{D}_{\lambda^{-1}}\widehat{T}, \varphi\rangle,$$

so $\widehat{\mathsf{D}_\lambda T} = \lambda^{-n}\mathsf{D}_{\lambda^{-1}}\widehat{T}$.

(c) Show that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is radially symmetric then so is $\widehat{T}$. Show that if T is α-homogeneous then $\widehat{T}$ is β-homogeneous for some $\beta \in \mathbb{R}$. What is β?

Solution: The first part is immediate. For the second part, if T is α-homogeneous,

$$\langle \mathsf{D}_\lambda\widehat{T}, \varphi\rangle = \langle \lambda^{-n}\widehat{\mathsf{D}_{\lambda^{-1}}T}, \varphi\rangle = \langle \lambda^{-n}\widehat{\lambda^{-\alpha}T}, \varphi\rangle = \lambda^{-n-\alpha}\langle \widehat{T}, \varphi\rangle \qquad \forall\varphi \in \mathcal{S},$$

hence $\mathsf{D}_\lambda\widehat{T} = \lambda^{-n-\alpha}\widehat{T}$ and $\widehat{T}$ is $(-n-\alpha)$-homogeneous.

(d) Show that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ and is radially symmetric (that is, $\mathsf{R}_A f = f$ for every $A \in \mathsf{SO}(n)$) and α-homogeneous (that is, $\mathsf{D}_\lambda f = \lambda^\alpha f\ \forall\lambda > 0$), then $f(x) = c|x|^\alpha$ almost everywhere, for some $c \in \mathbb{R}$.

Solution: Recall that two L^1_{loc} functions agree as distibutions if and only if they agree almost everywhere. Thus, we have that $\forall A \in \mathsf{SO}(n), \forall\lambda > 0$, almost every $z \in \mathbb{R}^n$ satisfies $f(z) = f(Az)$ and $f(\lambda z) = \lambda^\alpha f(z)$.

Fix $x, y \in \mathbb{R}^n \setminus \{0\}$ two Lebesgue points of f, and choose $\lambda > 0$ and $A \in \mathsf{SO}(n)$ such that $y = \lambda Ax$. Then we have that $\forall\varepsilon > 0$,

$$\begin{aligned}\frac{1}{\omega_n\varepsilon^n}\int_{B_\varepsilon(x)} f(z)\,\mathrm{d}z &= \frac{1}{\omega_n\varepsilon^n}\int_{B_\varepsilon(x)} f(A^{-1}z)\,\mathrm{d}z = \frac{1}{\omega_n\varepsilon^n}\int_{B_\varepsilon(Ax)} f(z')\,\mathrm{d}z' = \frac{\lambda^{-\alpha}}{\omega_n\varepsilon^n}\int_{B_\varepsilon(Ax)} f(\lambda z')\,\mathrm{d}z' \\ &= \frac{\lambda^{-\alpha-n}}{\omega_n\varepsilon^n}\int_{B_{\lambda\varepsilon}(\lambda Ax)} f(z'')\,\mathrm{d}z'' = \frac{\lambda^{-\alpha}}{\omega_n(\lambda\varepsilon)^n}\int_{B_{\lambda\varepsilon}(y)} f(z'')\,\mathrm{d}z''.\end{aligned}$$

Letting $\varepsilon \to 0$, since x and y are Lebesgue points we obtain that $f(x) = \lambda^{-\alpha}f(y) = \left(\frac{|y|}{|x|}\right)^{-\alpha} f(y)$. Fixing x and letting y vary, this shows that $f(y) = c|y|^\alpha$ for almost every $y \in \mathbb{R}^n$, where $c = f(x)/|x|^\alpha$.