Exercise 4.1. ★
Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$, and recall that we defined $T \star S, S \star T \in \mathcal{S}'(\mathbb{R}^n)$ as

$$\langle T \star S, \varphi \rangle = \langle T, \check{S} \star \varphi \rangle_{\mathcal{S}',\mathcal{S}} \qquad \text{and} \qquad \langle S \star T, \varphi \rangle = \langle S, \check{T} \star \varphi \rangle_{\mathcal{E}',C^\infty}$$

respectively, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(a) Let $(\chi_\varepsilon)_{\varepsilon>0}$ be a sequence of mollifiers as usual, and for $S \in \mathcal{E}'(\mathbb{R}^n)$, define $S_\varepsilon := S \star \chi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. Show that, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$S_\varepsilon \star \varphi \xrightarrow{\varepsilon \to 0} S \star \varphi \qquad \text{in } \mathcal{S}(\mathbb{R}^n).$$

Solution: We have that

$$S_\varepsilon \star \varphi(x) = \langle S_\varepsilon,\ \varphi(x-\cdot)\rangle = \langle S \star \chi_\varepsilon,\ \varphi(x-\cdot)\rangle = \langle S, \check{\chi}_\varepsilon \star (\varphi(x-\cdot))\rangle, = \langle S, (\chi_\varepsilon \star \varphi)(x-\cdot)\rangle,$$

because $(\check{\chi}_\varepsilon \star \varphi(x-\cdot))(y) = (\chi_\varepsilon \star \varphi)(x-y)$. Since $\partial^\alpha(S_\varepsilon \star \varphi) = S_\varepsilon \star \partial^\alpha \varphi$, it suffices to prove that $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ and any $\beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} \left| x^\beta \left(\langle S, (\chi_\varepsilon \star \varphi)(x-\cdot)\rangle - \langle S, \varphi(x-\cdot)\rangle \right) \right| \xrightarrow{\varepsilon \to 0} 0.$$

Let p be the order of S. Since $K := \operatorname{supp} S \subset B_\rho(0)$ is compact, there exists a constant $C_S > 0$,such that

$$|\langle S, \psi \rangle| \ \le C_S \sum_{|\alpha| \le p} \|\partial^\alpha \psi\|_{L^\infty(K)}, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

Hence we have for any $x \in \mathbb{R}^n$,

$$|x^\beta \left(\langle S, (\chi_\varepsilon \star \varphi)(x-\cdot)\rangle - \langle S, \varphi(x-\cdot)\rangle \right)| \le C_S \sum_{|\alpha| \le p} \|x^\beta \left((\chi_\varepsilon \star \partial^\alpha \varphi)(x-y) - \partial^\alpha \varphi(x-y) \right)\|_{L^\infty_y(K)}.$$

Let $\delta > 0$ and $R > 2\rho > 0$ to be fixed later on. For $x \in \mathbb{R}^n \setminus B_R(0)$ we bound

$$\left| x^\beta \left(\langle S, \chi_\varepsilon \star \varphi(x-\cdot)\rangle - \langle S, \varphi(x-\cdot)\rangle \right) \right| \le C_S \sum_{|\alpha| \le p} \|x^\beta \left((\chi_\varepsilon \star \partial^\alpha \varphi)(x-y) - \partial^\alpha \varphi(x-y) \right)\|_{L^\infty_y(B_\rho(0))}.$$

Observe that for $|x| > R > 2\rho$ and $y \in B_\rho(0)$ one has $2|x| > |x-y| > |x|/2$ and also

$$|(\chi_\varepsilon \star \partial^\alpha \varphi)(x-y)| \le \|\chi_\varepsilon\|_{L^1(\mathbb{R}^n)}\ \|\partial^\alpha \varphi\|_{L^\infty(B_{\rho+\varepsilon}(x))} = \|\partial^\alpha \varphi\|_{L^\infty(B_{\rho+\varepsilon}(x))} \le C|x|^{-1} \mathcal{N}_{p+1}(\varphi).$$

Hence we have for any $|\alpha| \le p$

$$C_S \sum_{|\alpha| \le p} \left\| \|x^\beta \left((\chi_\varepsilon \star \partial^\alpha \varphi)(x-y) - \partial^\alpha \varphi(x-y) \right)\|_{L^\infty_y(B_\rho(0))} \right\|_{L^\infty_x(\mathbb{R}^n \setminus B_R(0))} \quad \le C_p\ R^{-1}\ \mathcal{N}_{p+1}(\varphi)\ .$$

We choose R such that $C_p\ R^{-1}\ \mathcal{N}_{p+1}(\varphi) < \delta/2$. Being R now fixed, on $B_{R+\rho}(0)$ the convergence of $\chi_\varepsilon \star \partial^\alpha \varphi$ towards $\partial^\alpha \varphi$ is uniform, hence

$$\begin{aligned} C_S \sum_{|\alpha| \le p} &\left\| \|x^\beta \left((\chi_\varepsilon \star \partial^\alpha \varphi)(x-y) - \partial^\alpha \varphi(x-y) \right)\|_{L^\infty_y(B_\rho(0))} \right\|_{L^\infty_x(B_R(0))} \\ &\le C_S R^{|\beta|} \sum_{|\alpha| \le p} \|((\chi_\varepsilon \star \partial^\alpha \varphi)(z) - \partial^\alpha \varphi(z))\|_{L^\infty_z(B_{R+\rho}(0))} \le \delta/2 \end{aligned}$$

for ε small enough and we are done.

(b) Show that $S \star T = T \star S$ whenever $T \in \mathcal{S}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$. To do this, first prove it for S_ε in place of S, and then use the first part (applied to $\check{S}$) to conclude.

Solution: Let us prove this formula first for $S_\varepsilon \in C_c^\infty(\mathbb{R}^n)$:

$$\begin{aligned}\langle S_\varepsilon \star T, \varphi\rangle &= \langle S_\varepsilon, \check{T} \star \varphi\rangle = \int S_\varepsilon(x)(\check{T} \star \varphi)(x)\,\mathrm{d}x = \int S_\varepsilon(x)\langle \check{T}, \varphi(x-\cdot)\rangle\,\mathrm{d}x \\ &= \left\langle \check{T}, \int S_\varepsilon(x)\varphi(x-\cdot)\,\mathrm{d}x\right\rangle = \left\langle T, \int S_\varepsilon(x)\varphi(x+\cdot)\,\mathrm{d}x\right\rangle = \left\langle T, \check{S}_\varepsilon \star \varphi\right\rangle = \langle T \star S_\varepsilon, \varphi\rangle\,,\end{aligned}$$

where we have exchanged $\check{T}$ and the integral of Schwartz functions parametrized by x, and we also have used the equality

$$\int S_\varepsilon(x)\varphi(x+y)\,\mathrm{d}x = \int \check{S}_\varepsilon(-x)\varphi(x+y)\,\mathrm{d}x \overset{z=-x}{=} \int \check{S}_\varepsilon(z)\varphi(y-z)\,\mathrm{d}z = \check{S}_\varepsilon \star \varphi(y).$$

Now fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and observe that

$$\langle S_\varepsilon \star T, \varphi\rangle = \langle S_\varepsilon, \check{T} \star \varphi\rangle = \langle S, \chi_\varepsilon \star (\check{T} \star \varphi)\rangle \xrightarrow{\varepsilon\to 0} \langle S, (\check{T} \star \varphi)\rangle = \langle S \star T, \varphi\rangle$$

since $\chi_\varepsilon \star (\check{T} \star \varphi) \to \check{T} \star \varphi \in C^\infty(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$. On the other hand,

$$\langle T \star S_\varepsilon, \varphi\rangle = \langle T, \check{S}_\varepsilon \star \varphi\rangle \xrightarrow{\varepsilon\to 0} \langle T, \check{S} \star \varphi\rangle = \langle T \star S, \varphi\rangle$$

by part (a) and because $T \in \mathcal{S}'(\mathbb{R}^n)$. The result follows.

Exercise 4.2.
For each $0 < \alpha < n$, show that the function $f(x) = |x|^{-\alpha}$ defines a tempered distribution and compute its Fourier transform.
Hint: first consider $\alpha > n/2$, show that $\hat{f}$ is an L^1_{loc} function and apply Exercise 3.5 to deduce that $\hat{f}(\xi) = \gamma|\xi|^\beta$ for some $\beta, \gamma \in \mathbb{R}$ with β explicit. In order to find γ, test against a Gaussian $e^{-|x|^2/2}$, integrate in polar coordinates and relate the resulting expression to the Γ function. Argue for $\alpha < n/2$ using the inverse Fourier transform and finally for $\alpha = n/2$ by approximation.

Solution: If $n/2 < \alpha < n$, then $f(x) = |x|^{-\alpha} = |x|^{-\alpha}\chi_{\{|x|\le 1\}} + |x|^{-\alpha}\chi_{\{|x|>1\}} \in L^1 + L^2$, so $\hat{f}$ can be written as the sum of a continuous function vanishing at infinity (and hence in L^∞) and an L^2 function, and therefore $\hat{f} \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Since f is radially symmetric and $(-\alpha)$-homogeneous (as a distribution), $\hat{f}$ is radially symmetric and $(-n+\alpha)$-homogeneous, and using part (d) and the fact that $\hat{f} \in L^1_{\mathrm{loc}}$ we get that $\hat{f}(\xi) = \gamma_{n,\alpha}|\xi|^{-(n-\alpha)}$ for a constant $\gamma_{n,\alpha} \in \mathbb{C}$.

In order to find $\gamma_{n,\alpha}$, we test against $g(x) = e^{-|x|^2/2}$, with $\hat{g}(\xi) = e^{-|\xi|^2/2}$:

$$\int_{\mathbb{R}^n} |x|^{-\alpha} e^{-|x|^2/2}\,\mathrm{d}x = \gamma_{n,\alpha}\int_{\mathbb{R}^n} |\xi|^{-(n-\alpha)} e^{-|\xi|^2/2}\,\mathrm{d}\xi.$$

On the left we have, using the change of variables $s = r^2/2$,

$$\int_{\mathbb{R}^n} |x|^{-\alpha} e^{-|x|^2/2}\,\mathrm{d}x = \int_0^\infty r^{-\alpha} e^{-r^2/2} n\omega_n r^{n-1}\,\mathrm{d}r = n\omega_n \int_0^\infty (2s)^{\frac{n-\alpha}{2}-1} e^{-s}\,\mathrm{d}s$$
$$= n\omega_n 2^{\frac{n-\alpha}{2}-1}\Gamma\left(\frac{n-\alpha}{2}\right)$$

and similarly

$$\int_{\mathbb{R}^n} |\xi|^{-(n-\alpha)} e^{-|\xi|^2/2}\,\mathrm{d}\xi = n\omega_n 2^{\frac{n-(n-\alpha)}{2}-1}\Gamma\left(\frac{n-(n-\alpha)}{2}\right) = n\omega_n 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right),$$

from which we deduce

$$\gamma_{n,\alpha} = \frac{n\omega_n 2^{\frac{n-\alpha}{2}-1}\Gamma\left(\frac{n-\alpha}{2}\right)}{n\omega_n 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)} = 2^{\frac{n}{2}-\alpha}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

Thanks to the Fourier inversion formula, we get for free that the Fourier transform of $|x|^{-\alpha}$ is $\gamma_{n,n-\alpha}^{-1}|\xi|^{-(n-\alpha)}$ for $0 < \alpha < n/2$. Finally, for $\alpha = n/2$, choose a sequence $\alpha_j \to \alpha$ and use the fact that $|x|^{-\alpha_j} \rightharpoonup |x|^{-\alpha}$ and $|\xi|^{-(n-\alpha_j)} \rightharpoonup |\xi|^{-(n-\alpha)}$ in $\mathcal{S}'$ (using the dominated convergence theorem when we test against $\varphi \in \mathcal{S}$), together with the continuity of the Fourier transform with respect to weak convergence of tempered distributions.

Exercise 4.3.
Show that, for each $n \geq 2$ and $1 \leq i \leq n$, $\mathrm{p.v.}\,\dfrac{x_i}{|x|^{n+1}}$ defines a tempered distribution and compute its Fourier transform.
Hint: use the previous exercise.

Solution: We claim that $\mathrm{p.v.}\,\dfrac{x_i}{|x|^{n+1}} = \dfrac{1}{1-n}\partial_{x_i}|x|^{1-n}$, where $|x|^{1-n} \in \mathcal{S}'(\mathbb{R}^n)$ and we are taking the distributional derivative. Indeed, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\{|x|>\varepsilon\}} \frac{x_i}{|x|^{n+1}}\varphi(x)\,\mathrm{d}x &= \frac{1}{1-n}\int_{\{|x|>\varepsilon\}} \partial_{x_i}|x|^{1-n}\varphi(x)\,\mathrm{d}x = \frac{1}{1-n}\int_{\{|x|>\varepsilon\}} \mathrm{div}(e_i|x|^{1-n})\varphi(x)\,\mathrm{d}x\\ &= \int_{\partial B_\varepsilon} |x|^{1-n}\varphi(x) e_i\cdot\nu(x)\,\mathrm{d}\sigma(x) - \frac{1}{1-n}\int_{\{|x|>\varepsilon\}} |x|^{1-n} e_i\cdot\nabla\varphi(x)\,\mathrm{d}x\\ &= \varepsilon^{1-n}\int_{\partial B_\varepsilon} (\varphi(x)-\varphi(0))\frac{x_i}{|x|}\,\mathrm{d}\sigma(x) - \frac{1}{1-n}\int_{\{|x|>\varepsilon\}} |x|^{1-n}\partial_{x_i}\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Here we have subtracted $\varphi(0)$ thanks to the fact that $x_i/|x|$ is odd. The first term is bounded in absolute value by

$$\varepsilon^{1-n}\int_{\partial B_\varepsilon} |\varphi(x)-\varphi(0)|\,\mathrm{d}\sigma(x) \leq \varepsilon^{1-n}\cdot\varepsilon\cdot C\varepsilon^{n-1} \xrightarrow{\varepsilon\to 0} 0,$$

whereas the fact that $|x|^{1-n}$ is integrable around the origin allows us to pass to the limit the second term:

$$-\frac{1}{1-n}\int_{\{|x|>\varepsilon\}}|x|^{1-n}\partial_{x_i}\varphi(x)\,\mathrm{d}x \xrightarrow{\varepsilon\to 0} -\frac{1}{1-n}\langle|x|^{1-n},\partial_{x_i}\varphi\rangle = \frac{1}{1-n}\left\langle\partial_{x_i}|x|^{1-n},\varphi\right\rangle.$$

This proves the claim. Denote $f(x)=|x|^{1-n}$. We have by Exercise 3.?(e) that $\hat{f}(\xi)=\gamma|\xi|^{-1}$ with

$$\gamma=\gamma_{n,n-1}=2^{1-\frac{n}{2}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

It follows that

$$\widehat{\mathrm{p.\,v.}\,\frac{x_i}{|x|^{n+1}}}(\xi)=\frac{1}{1-n}\widehat{\partial_{x_i}|x|^{1-n}}(\xi)=\frac{1}{1-n}i\xi_i\widehat{f}(\xi)=i\frac{\gamma}{1-n}\frac{\xi_i}{|\xi|}.$$

Exercise 4.4. ★
Recall that the distribution $S\in\mathcal{S}'(\mathbb{R}^4)$ defined by

$$\langle S,\varphi\rangle := \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\varphi(x,|x|)}{|x|}\,\mathrm{d}x,\qquad \varphi\in\mathcal{S}(\mathbb{R}^4),$$

is a fundamental solution of the wave operator $\square$. Show that, given $f\in\mathcal{E}'(\mathbb{R}^4)$, $u:=S\star f$ is the only solution to $\square u=f$ which is supported in $\mathbb{R}^3\times(t_0,\infty)$ for some $t_0>0$.

Solution: Assume there exists another solution $\tilde{u}\in\mathcal{S}'(\mathbb{R}^4)$ supported in $\mathbb{R}^3\times(t_0',+\infty)$ for some t_0'. Denoting $w:=u-\tilde{u}$, which is supported in $\mathbb{R}^3\times(t_0'',+\infty)$ and satisfies $\square w=0$, we have

$$w=\delta_0\star w=(S\star\square\,\delta_0)\star w.$$

Let now $\Theta\in C_c^\infty(B_2(0))$ with $\Theta\equiv 1$ on $B_1(0)$, and define $\Theta_i(x)=\Theta(x/i),\ i\in\mathbb{N}$. Then $\Theta_i\equiv 1$ on $B_i(0)$ and $\Theta_i\equiv 0$ on $B_{2i}(0)^c$. This gives

$$(\underbrace{\Theta_i S}_{\in\mathcal{E}'}\star\underbrace{\square\,\delta_0}_{\in\mathcal{E}'})\star\underbrace{w}_{\in\mathcal{S}'}=\Theta_i S\star(\square\,\delta_0\star w)=\Theta_i S\star\square\,w=0=\Theta_i S\star 0=0. \tag{1}$$

Moreover there holds

$$\square\,(\Theta_i S)=\square S=\delta_0\qquad\text{in }\mathcal{D}'(B_i(0))\ ,$$

and

$$\square\,(\Theta_i S)=0\qquad\text{in }\mathcal{D}'(B_{2i}(0)^c).$$

Thus $\square\,(\Theta_i S)=\delta_0+h_i$ for a distribution h_i supported in $B_{2i}(0)\setminus B_i(0)\cap\Lambda^+$, where $\Lambda^+=\operatorname{supp}S$ denotes the positive light cone. Let now $\varphi\in C_c^\infty(\mathbb{R}^4)$ with $\operatorname{supp}\varphi\subset B_R^4(0)$. Since

$$\langle(\Theta_i S\star\square\delta_0)\star w,\,\varphi\rangle=\langle\square\Theta_i S,\,\check{w}\star\varphi\rangle \tag{2}$$

and

$$\operatorname{supp} \check{w} \subset \{(x,t),\, t < -t_0''\},$$

this implies

$$\operatorname{supp}(\check{w} \star \varphi) \subset \{(x,t), t \le -t_0'' + R\}\ .$$

On the other hand, $\operatorname{supp} h_i \subseteq \mathbb{R}^4 \setminus B_i \cap \Lambda^+ = \{(x,t) : t = |x| > i/\sqrt{2}\}$. Thus

$$\langle h_i, \check{w} \star \varphi \rangle_{\mathcal{E}', C^\infty} = 0$$

for i large enough. Combining the above we have for i large enough

$$\begin{aligned} 0 \underset{\substack{\uparrow \\ (1),(2)}}{=} \left\langle \square\, \Theta_i \tfrac{T}{4\pi\rho},\, \check{w} \star \varphi \right\rangle &= \langle \delta_0 + h_i,\, \check{w} \star \varphi \rangle \\ = \langle \delta_0, \check{w} \star \varphi \rangle &= \langle \delta_0, \langle w(-y),\, \varphi(x-y) \rangle \rangle = \langle w, \varphi \rangle. \end{aligned}$$

Hence we have proved that $w = u - \tilde{u} = 0$ in $\mathcal{D}'(\mathbb{R}^4)$. This holds as well in $\mathcal{S}'(\mathbb{R}^4)$ since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

Exercise 4.5. ★

(a) Show that the formal solution to the heat equation with initial data $f \in \mathcal{S}'(\mathbb{R}^n)$ obtained in the lecture,

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \left(e^{-|\cdot|^2/4t} \star f \right)(x) \tag{†}$$

satisfies the initial condition in the following sense:

$$u(t,\cdot) \xrightarrow{t \to 0} f \qquad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{IC}$$

Solution: Recall that, in Fourier, the solution is given by

$$\hat{u}(t) = e^{-t|\xi|^2} \hat{f},$$

where we are multiplying $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$ with a Gaussian, which is a Schwartz function. We need to prove that, given $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \hat{f}, e^{-t|\xi|^2} \psi \rangle = \langle e^{-t|\xi|^2} \hat{f}, \psi \rangle \xrightarrow{t \to 0} \langle \hat{f}, \psi \rangle,$$

so it suffices to show that $e^{-t|\xi|^2}\psi \to \psi$ in $\mathcal{S}$. First of all, when we do not take any derivatives, we have to prove that given $m \in \mathbb{N}$ and $\varepsilon > 0$, for t small enough,

$$|\xi|^m (1 - e^{-t|\xi|^2}) |\psi(\xi)| \le \varepsilon \qquad \forall \xi \in \mathbb{R}^n. \tag{1}$$

Let $R > 0$ so that $|\xi|^m |\psi|(\xi) \leq \varepsilon$ for all ξ with $|\xi| > R$. Then clearly (1) holds on $\mathbb{R}^n \setminus B_R$ for every t, and it also holds on B_R for t small enough thanks to the fact that $1 - e^{-t|\xi|^2} \leq 1 - e^{-tR^2} \to 0$.

To prove convergence in the Schwartz space, we also have to show that for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$, $|\xi|^m \partial^\alpha \Big((1 - e^{-t|\xi|^2})\psi(\xi)\Big) \to 0$ uniformly in $\mathbb{R}^n$. One can easily see by induction that for every such α,

$$|\xi|^m \partial^\alpha \Big((1 - e^{-t|\xi|^2})\psi(\xi)\Big) = |\xi|^m (1 - e^{-t|\xi|^2}) \partial^\alpha \psi(\xi) + |\xi|^m \sum_{|\beta| \leq m} t\, p_\beta(t, \xi) e^{-t|\xi|^2} \partial^\beta \psi(\xi),$$

where the p_β are polynomials in t and ξ coming from differentiating the exponential. We have already shown uniform convergence of the first summand. For the terms in the second sum we first bound, for $t \leq 1$, $p_\beta(t, \xi) \leq C(1 + |\xi|^k)$ for some k, and then

$$t|\xi|^m \, |p_\beta(t, \xi)| e^{-t|\xi|^2} |\partial^\beta \psi(\xi)| \leq t|\xi|^m \,(1 + |\xi|^k) |\partial^\beta \psi(\xi)| \leq C \mathcal{N}_{2m+k}(\psi) t \xrightarrow{t \to 0} 0.$$

(b) Show rigorously that, if $u \in C^1(\mathbb{R}^+, L^1(\mathbb{R}^n))$ satisfies (IC) for some $f \in L^1(\mathbb{R}^n)$ and also

$$\partial_t \langle u, \varphi \rangle = \langle \Delta u, \varphi \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n),$$

then u must be given by the formula (†) and in particular it is unique in this class.

Solution: Let $\hat{u}(t, x)$ be the Fourier transform of u in space; since $u(t, \cdot) \in L^1$ for each t, it follows that $\hat{u}(t, \cdot)$ is continuous for each t; the continuous differentiability in t imply that $t \in \mathbb{R}^+ \mapsto \hat{u}(t, \cdot) \in C^0(\mathbb{R}^n)$ is also C^1, and in particular $\partial_t \hat{u}$ can be computed pointwise.
Fourier-transforming the equation, we get that

$$\langle \partial_t \hat{u}, \psi \rangle = \partial_t \langle \hat{u}, \psi \rangle = -\langle |\xi|^2 \hat{u}, \psi \rangle \qquad \forall \psi \in \mathcal{S}(\mathbb{R}^n),$$

so since $\hat{u}$ and $\partial_t \hat{u}$ are continuous we get that $\partial_t \hat{u}(t, \xi) = -|\xi|^2 \hat{u}(t, \xi)$ pointwise for all $t > 0$ and $\xi \in \mathbb{R}^n$. Solving the ODE we have that $\hat{u}(t, \xi) = c(\xi) e^{-t|\xi|^2}$ for some constant $c(\xi)$ such that $\lim_{t \to 0} \hat{u}(t, \xi) = c(\xi)$. The initial condition becomes also

$$\langle c(\cdot), \psi \rangle = \lim_{t \to 0} \langle \hat{u}(t, \cdot), \psi \rangle = \langle \hat{f}, \psi \rangle,$$

from which it follows that $\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{f}(\xi)$. The equation (†) now follows from the formula for the convolution, and this determines u uniquely.