Exercise 6.1.
Prove the Heisenberg uncertainty principle: given $f \in L^2(\mathbb{R}, \mathbb{C})$, show that

$$\left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 \, \mathrm{d}x\right) \left(\int_{\mathbb{R}} |\xi|^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right) \geq \frac{1}{4} \left(\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x\right)^2 = \frac{1}{4} \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^2.$$

For which functions does equality hold?

Hint: consider the integral $\int_{-\infty}^{\infty} x \overline{f(x)} f'(x) \, \mathrm{d}x$.

Solution: Of course we can assume that $f \in H^1(\mathbb{R})$, otherwise the left hand side is infinite. We integrate by parts

$$\int_{\mathbb{R}} x \overline{f(x)} f'(x) \, \mathrm{d}x = -\int_{\mathbb{R}} (x \overline{f(x)})' f(x) \, \mathrm{d}x = -\int_{\mathbb{R}} \overline{f(x)} f(x) \, \mathrm{d}x - \int_{\mathbb{R}} x \overline{f'(x)} f(x) \, \mathrm{d}x.$$

Thus

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = -\int_{\mathbb{R}} x \left(\overline{f(x)} f'(x) + \overline{f'(x)} f(x)\right) \, \mathrm{d}x = -2 \operatorname{Re} \int_{\mathbb{R}} x \overline{f(x)} f'(x) \, \mathrm{d}x.$$

Now using Cauchy–Schwarz we get

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq 2 \left(\int_{\mathbb{R}} |x \overline{f(x)}|^2 \, \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}} |f'(x)|^2 \, \mathrm{d}x\right)^{1/2}.$$

The inequality follows by using Plancherel and the formula $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$.

The equality case follows from the equality in Cauchy–Schwarz: there must exist $\lambda > 0$ such that $f'(x) = -\lambda x f(x)$ for almost every x. We would like to integrate this ODE and see that $f(x) = ce^{-\lambda x^2/2}$, but unfortunately we do not have enough regularity to do so. However we can compute the distributional derivative of $f(x)e^{\lambda x^2/2}$ and see that it vanishes: let $\varphi \in C_c^\infty(\mathbb{R})$, then for almost every x we have

$$\begin{aligned} f(x)e^{\lambda x^2/2}\varphi'(x) &= f(x)\left(e^{\lambda x^2/2}\varphi(x)\right)' - \lambda x f(x) e^{\lambda x^2/2}\varphi(x) \\ &= f(x)\left(e^{\lambda x^2/2}\varphi(x)\right)' + f'(x)e^{\lambda x^2/2}\varphi(x) = \left(f(x)e^{\lambda x^2/2}\varphi(x)\right)'. \end{aligned}$$

Then integrating and using the fact that φ has compact support, we get

$$\int_{\mathbb{R}} f(x)e^{\lambda x^2/2}\varphi'(x) \, \mathrm{d}x = \int_{\mathbb{R}} \left(f(x)e^{\lambda x^2/2}\varphi(x)\right)' \, \mathrm{d}x = 0,$$

which means that $f(x)e^{\lambda x^2/2}$ has zero distributional derivative, and hence (as we saw in the lecture) it is constant. Thus f is a Gaussian $f(x) = ce^{-\lambda x^2/2}$ (almost everywhere).

Exercise 6.2.

(a) Show that every $u \in H^{-1}(\mathbb{R})$ can be expressed as $f + g'$ for some functions $f, g \in L^2(\mathbb{R})$ (here the prime denotes the distributional derivative).

Solution: We know that

$$\|u\|_{H^{-1}}^2 = \int_{\mathbb{R}} |\hat{u}|^2(\xi) \frac{1}{1+\xi^2} \,\mathrm{d}\xi = \int_{[-1,1]} |\hat{u}|^2(\xi) \frac{1}{1+\xi^2} \,\mathrm{d}\xi + \int_{\mathbb{R}\setminus[-1,1]} \frac{|\hat{u}|^2(\xi)}{|\xi|^2} \frac{|\xi|^2}{1+\xi^2} \,\mathrm{d}\xi$$

is finite. Thus letting

$$\hat{f}(\xi) := \hat{u}(\xi) \mathbb{1}_{[-1,1]}(\xi) \qquad \text{and} \qquad \hat{g}(\xi) := \frac{\hat{u}(\xi)}{i\xi} \mathbb{1}_{\mathbb{R}\setminus[-1,1]}(\xi),$$

the above expression together with the bounds

$$\frac{1}{1+\xi^2} \geq \frac{1}{2} \quad \text{for } \xi \in [-1,1], \qquad\qquad \frac{\xi^2}{1+\xi^2} \geq \frac{1}{2} \quad \text{for } \xi \in \mathbb{R} \setminus [-1,1]$$

shows that $\hat{f}, \hat{g} \in L^2(\mathbb{R})$ and hence they are the Fourier transforms of functions $f, g \in L^2(\mathbb{R})$. It is then immediate to check that

$$\widehat{f+g'}(\xi) = \hat{f}(\xi) + i\xi\hat{g}(\xi) = \hat{u}(\xi)\mathbb{1}_{[-1,1]}(\xi) + \hat{u}(\xi)\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(\xi) = \hat{u}(\xi),$$

hence $f + g' = u$ in the sense of distributions.

(b) Deduce that for every $u \in H^{-1}(\mathbb{R})$ there exists a function $v \in L^2_{\mathrm{loc}}(\mathbb{R})$ such that $v' = u$ in the distributional sense.

Solution: Write $u = f + g'$ as above. Let

$$v(x) := g(x) + \int_0^x f(y) \,\mathrm{d}y \qquad \text{for almost every } x \in \mathbb{R},$$

where we interpret $\int_0^x f(y) \,\mathrm{d}y$ as $-\int_x^0 f(y) \,\mathrm{d}y$ in case $x < 0$.

It is easy to see that $x \mapsto \int_{[0,x]} f$ is continuous—in fact, $C^{1/2}$ thanks to Cauchy–Schwarz, therefore the second function is L^2_{loc} and thus $v \in L^2_{\mathrm{loc}}(\mathbb{R})$ as well. To see that the distributional derivative of the second term is f, choose $R > 0$ and set $f_R := f\mathbb{1}_{[-R,R]}$. Let also $H(y) := \mathbb{1}_{[0,\infty)}(y)$. Then for $x \in (-R, R)$ it holds

$$f_R \star H(x) = \int_{\mathbb{R}} f_R(y) H(x-y) \,\mathrm{d}y = \int_{-\infty}^x f_R(y) \,\mathrm{d}y = \int_{-R}^x f(y) \,\mathrm{d}y = \int_{-R}^0 f(y) \,\mathrm{d}y + \int_0^x f(y) \,\mathrm{d}y.$$

Taking distributional derivatives we then have

$$\left(\int_0^x f(y) \,\mathrm{d}y\right)' = \left(\int_{-R}^0 f(y) \,\mathrm{d}y + \int_0^x f(y) \,\mathrm{d}y\right)' = (f_R \star H)'(x) = f_R \star \delta(x) = f_R(x) = f(x).$$

Exercise 6.3. ♣

The goal of this exercise is to reflect about the meaning of the dual of the spaces H^s in terms of abstract functional analysis.

(a) Let V be a finite-dimensional vector space, and let $W \subsetneq V$ be a subspace. Recall that the inclusion $i : W \to V$ induces a dual map $i^* : V^* \to W^*$, "the restriction to W of linear functionals on V", and that i^* is *surjective* and therefore *never injective*.

Now consider two reflexive Banach spaces V, W and a continuous embedding $i : W \to V$ such that $i(W) \subsetneq V$ but $i(W)$ is dense in V. Prove that this map induces a continuous linear map $i^* : V^* \to W^*$, and that i^* is *injective* but *never surjective* (!).

Think about what this means in the case $V = L^2(\mathbb{R}^n)$ and $W = H^s(\mathbb{R}^n)$ for $s > 0$.

Solution: Recall that the induced map $i^* : V^* \to W^*$ is given by

$$i^*(T)(w) = T(i(w)) \qquad \text{for } T \in V^* \text{ and } w \in W.$$

To see injectivity, suppose that $i^*(T) = 0$, so that $T(i(w)) = 0$ for every $w \in W$. Since $i(W)$ is dense in V and T is continuous in V, it follows that T vanishes on all of V.

To see that this map is not surjective, we argue by contradiction. Assuming that i^* is surjective, since it is injective, the open mapping theorem gives us an inverse map $E : W^* \to V^*$ (that we can think of as extending linear functionals). Dualizing we obtain a map $E^* : V^{**} \to W^{**}$. Now let $v \in V \setminus i(W)$ and use the reflexivity of W to obtain a vector $w \in W$ such that $E^*(\Phi_v) = \Phi_w$, where $\Phi_{(\cdot)}$ denotes respectively the canonical embedding of V into V^{**} and of W into W^{**}.

We claim that $i(w) = v$, which is a contradiction. To see that, by Hahn–Banach's theorem it is enough to check that for every $T \in V^*$, $T(i(w)) = T(v)$. But we have indeed

$$\langle T, v\rangle = \langle E(i^*(T)), v\rangle = \langle \Phi_v, E(i^*(T))\rangle = \langle E^*(\Phi_v), i^*(T)\rangle = \langle \Phi_w, i^*(T)\rangle = \langle i^*(T), w\rangle = \langle T, i(w)\rangle$$

and the claim follows.

(b) We know that $H^s(\mathbb{R}^n)$ is a Hilbert space and therefore it is canonically isomorphic to its dual, but we have learned that the dual of $H^s(\mathbb{R}^n)$ is $H^{-s}(\mathbb{R}^n)$. Make these statements precise in order to avoid a contradiction.

Solution: The isomorphism between the dual of H^s and H^s given by Riesz's representation theorem is with respect to the scalar product

$$\langle f, g\rangle_{H^s} = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1 + |\xi|^2)^s \, \mathrm{d}\xi,$$

which is slightly artificial.

When we say that the dual of H^s is H^{-s} we are using the L^2 scalar product, and the precise claim is that the bilinear form $\langle \cdot, \cdot\rangle_{L^2} : C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ extends to a pairing $\langle \cdot, \cdot\rangle : H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \to \mathbb{C}$ which realizes the dual of each space, that is,

$$\forall\, T \in H^s(\mathbb{R}^n)^* \quad \exists!\, v \in H^{-s}(\mathbb{R}^n) \quad \text{s.t.} \quad T(u) = \langle u, v\rangle \quad \forall\, u \in H^s(\mathbb{R}^n)$$

and

$$\forall\, S \in H^{-s}(\mathbb{R}^n)^* \quad \exists!\, u \in H^s(\mathbb{R}^n) \quad \text{s.t.} \quad S(v) = \langle u, v\rangle \quad \forall\, v \in H^{-s}(\mathbb{R}^n).$$

Exercise 6.4.
Show that every finite Radon measure μ on $\mathbb{R}^n$ induces a distribution in $H^\sigma(\mathbb{R}^n)$ for each $\sigma < -\frac{n}{2}$.

Solution: It is enough to show that the operator $\varphi \in C_c^\infty(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu$ extends continuously to H^s for each $s > \frac{n}{2}$, and by density it is enough to show the estimate

$$\left| \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \right| \leq C \|\varphi\|_{H^s(\mathbb{R}^n)} \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

But this follows immediately from the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ for $s > \frac{n}{2}$.

Exercise 6.5. ★
In this exercise we will show that, for every $n \geq 1$, there is a function $u \in H^{n/2}(\mathbb{R}^n)$ which is not in $L^\infty(\mathbb{R}^n)$.

(a) Construct a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

- $f > 0$ everywhere;
- $f \notin L^1(\mathbb{R}^n)$;
- $\int_{\mathbb{R}^n} |f(\xi)|^2 (1 + |\xi|^2)^{n/2} \, \mathrm{d}\xi < +\infty$.

Hint: use the Ansatz $f(\xi) = (1 + |\xi|^2)^{-n/2} g(|\xi|)$ and find a suitable function g.

Solution: Using the Ansatz from the hint, we need to find $g > 0$ with

$$\int_0^\infty g(r)(1 + r^2)^{-n/2} r^{n-1} \, \mathrm{d}r = +\infty \qquad \text{and} \qquad \int_0^\infty g(r)^2 (1 + r^2)^{-n/2} r^{n-1} \, \mathrm{d}r < +\infty.$$

We assume that g is bounded in $[0, 1]$ so that the two above requirements simplify to

$$\int_1^\infty g(r) \frac{\mathrm{d}r}{r} = +\infty \qquad \text{and} \qquad \int_1^\infty g(r)^2 \frac{\mathrm{d}r}{r} < +\infty.$$

Using the change of variable $r = e^t$ this becomes

$$\int_0^\infty g(e^t) \, \mathrm{d}t = +\infty \qquad \text{and} \qquad \int_0^\infty g(e^t)^2 \, \mathrm{d}t < +\infty,$$

and for example $g(e^t) = \frac{1}{1+t}$ works, giving $g(r) = \frac{1}{1+\log r} \mathbb{1}_{[1,\infty)}(r)$.

(b) Let u be defined by $\hat{u} = f$. Show that $u \in H^{n/2}(\mathbb{R}^n)$ and $u \notin L^\infty(\mathbb{R}^n)$.

Hint: try to capture the L^1 norm of f by testing against very spread out functions.

Solution: It is clear that $u \in H^{n/2}(\mathbb{R}^n)$. Fix $\varphi(\xi) := e^{-|\xi|^2/2}$ and, for $\lambda > 0$, let $\varphi_\lambda(\xi) := \mathsf{D}_\lambda \varphi(\xi) = \varphi(\lambda \xi)$. We know that $\widehat{\varphi_\lambda}(x) = \lambda^{-n} \widehat{\varphi}(\lambda^{-1} x)$ and therefore

$$\|\widehat{\varphi_\lambda}\|_{L^1(\mathbb{R}^n)} = \lambda^{-n} \int_{\mathbb{R}^n} \left| \widehat{\varphi}\left(\frac{x}{\lambda} \right) \right| \mathrm{d}x = \int_{\mathbb{R}^n} |\widehat{\varphi}(y)| \, \mathrm{d}y = \|\widehat{\varphi}\|_{L^1(\mathbb{R}^n)} = C$$

for some constant C. Hence

$$\langle f, \varphi_\lambda \rangle = \langle u, \widehat{\varphi_\lambda} \rangle \leq \|u\|_{L^\infty} \|\widehat{\varphi_\lambda}\|_{L^1} \leq C\|u\|_{L^\infty} = C'$$

but by the monotone convergence theorem, this implies that

$$\int_{\mathbb{R}^n} f(\xi)\,\mathrm{d}\xi = \lim_{\lambda \to 0} \int_{\mathbb{R}^n} f(\xi)\varphi_\lambda(\xi)\,\mathrm{d}\xi \leq C',$$

a contradiction.

Exercise 6.6.
Show that for every $n \geq 1$ there is no trace operator $T : H^{1/2}(\mathbb{R}^n_+) \to L^2(\mathbb{R}^{n-1})$.
Hint: think about the estimate that this entails for $n = 1$.

Solution: Let $u \in H^{1/2}(\mathbb{R})$ be the function produced in the previous exercise (for $n = 1$) which is not in $L^\infty(\mathbb{R})$, and let $w \in C^\infty_c(\mathbb{R}^{n-1})$ be some nonzero function. We consider $v(x) = v(x', x_n) := w(x')u(x_n)$, with $\hat{v}(\xi) = \hat{u}(\xi_n)\hat{w}(\xi')$. Using the smoothness of w, have that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\hat{v}|^2(\xi)(1+|\xi|^2)^{1/2}\,\mathrm{d}\xi &= \int_{\mathbb{R}^n} |\hat{u}|^2(\xi_n)|\hat{w}|^2(\xi')(1+|\xi'|^2+|\xi_n|^2)^{1/2}\,\mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}^n} |\hat{u}|^2(\xi_n)\frac{1+|\xi'|+|\xi_n|}{(1+|\xi'|)^{n+1}}\,\mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}^n} |\hat{u}|^2(\xi_n)\frac{(1+|\xi'|)(1+|\xi_n|)}{(1+|\xi'|)^{n+1}}\,\mathrm{d}\xi \\
&\leq C\int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\xi'}{(1+|\xi'|)^n}\int_{\mathbb{R}} (1+|\xi_n|^2)^{1/2}|\hat{u}|^2(\xi_n)\,\mathrm{d}\xi_n < +\infty.
\end{aligned}$$

On the other hand, assume that there is such a trace operator. Hence for smooth functions $f \in C^\infty_c(\mathbb{R}^n)$ we would have

$$\int_{\mathbb{R}^{n-1}} f(x',0)\varphi(x')\,\mathrm{d}x' \leq C\|f\|_{H^{1/2}(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^{n-1})} \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^{n-1}).$$

Applying this to $f(\cdot, t+\cdot)$ and using Fubini, for every $\psi \in C^\infty_c(\mathbb{R})$ we would have

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x', x_n)\varphi(x')\psi(x_n)\,\mathrm{d}x &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} f(x',t)\varphi(x')\,\mathrm{d}x' \right) \psi(t)\,\mathrm{d}t \\
&\leq C\|f\|_{H^{1/2}(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^{n-1})}\|\psi\|_{L^1(\mathbb{R})} \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^{n-1}),
\end{aligned}$$

and since both sides of the equation are continuous in $H^{1/2}$, by approximation we would get

$$\int_{\mathbb{R}^n} v(x', x_n)\varphi(x')\psi(x_n)\,\mathrm{d}x \leq C\|v\|_{H^{1/2}(\mathbb{R}^n)}\|\varphi\|_{L^2(\mathbb{R}^{n-1})}\|\psi\|_{L^1(\mathbb{R})} \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^{n-1})\ \forall \psi \in C^\infty_c(\mathbb{R}).$$

Choosing $\varphi = w$ above we get the estimate

$$\int_{\mathbb{R}} u(x_n)\psi(x_n)\,\mathrm{d}x_n \leq C\|\psi\|_{L^1(\mathbb{R})} \qquad \forall \psi \in C^\infty_c(\mathbb{R}),$$

and now this extends by continuity to all $\psi \in L^1(\mathbb{R})$ and shows that $u \in L^\infty(\mathbb{R})$, a contradiction.