Exercise 7.1.
Let V be a Banach space, $W \subset V$ a closed linear subspace and $v \in V \setminus W$. Show that there exists a linear functional $\ell \in V^*$ such that $\langle \ell, v\rangle \neq 0$ but $\langle \ell, w\rangle = 0$ for all $w \in W$. Moreover, show that one can choose ℓ such that $\langle \ell, v\rangle = \operatorname{dist}(v, W)$ and $\|\ell\| \leq 1$.

Solution: Let $d := \operatorname{dist}(v, W)$ and define the linear map $\ell_0 : W + \mathbb{R}v \to \mathbb{R}$ by $w + \lambda v \mapsto \lambda d$. This is well defined since $v \notin W$, and moreover, for $\lambda \neq 0$,

$$\|w + \lambda v\| = |\lambda| \left\| \frac{w}{\lambda} + v \right\| \geq |\lambda| \operatorname{dist}(v, W) = |\lambda d| = |\langle \ell_0, w + \lambda v\rangle|,$$

so that $\|\ell_0\| \leq 1$. Applying Hahn–Banach we get an extension $\ell : V \to \mathbb{R}$ with $\|\ell\| \leq 1$ still vanishing on W. Since W is closed, $\operatorname{dist}(v, W) > 0$ and hence $\langle \ell, v\rangle = d > 0$.

Exercise 7.2.
Prove that any closed linear subspace of a reflexive Banach space is also reflexive.

Solution: Let W be a closed linear subspace of a Banach space V, and let $i : W \to V$ denote the natural inclusion. Then we get a dual map $i^* : V^* \to W^*$, namely the restriction to W. Given $\phi \in W^{**}$, consider $i^{**}\phi \in V^{**}$, namely the composition $\phi \circ i^*$. Since V is reflexive we have that there exists $v \in V$ such that for all $\ell \in V^*$, $\langle i^{**}\phi, \ell\rangle = \langle \ell, v\rangle$.

We claim that $v \in W$: otherwise, Exercise 7.1 gives us a linear map $\ell \in V^*$ such that $\langle \ell, v\rangle \neq 0$ but $\langle \ell, w\rangle = 0$ for all $w \in W$. This last condition means that $i^*\ell = 0$, whence

$$\langle \ell, v\rangle = \langle i^{**}\phi, \ell\rangle = \langle \phi, i^*\ell\rangle = 0,$$

a contradiction. In order to show that this $v \in W$ is the desired vector, we first need to show that $i^* : V^* \to W^*$ is surjective: given a linear bounded functional $\vartheta : W \to \mathbb{R}$, the Hahn–Banach theorem allows us to extend it to some $\ell : V \to \mathbb{R}$, and in particular $i^*\ell = \vartheta$. Finally, for such $\vartheta \in W^*$, we have that $\langle \phi, \vartheta\rangle = \langle \phi, i^*\ell\rangle = \langle i^{**}\phi, \ell\rangle = \langle \ell, v\rangle = \langle \vartheta, v\rangle$ and we are done.

Exercise 7.3.
Let $(E, \|\cdot\|_E)$ be a Banach space. Prove that E is reflexive if and only if E^* is reflexive.

Solution: Suppose first that E is reflexive, so that the natural embedding $J_E : E \to E^{**}$ is surjective and hence an isomorphism. This gives us an isomorphism $J_E^* : E^{***} \to E^*$. We claim that the inverse of this isomorphism is $J_{E^*} : E^* \to E^{***}$: given $x \in E$ and $\ell \in E^*$, we have that

$$\langle J_E^* J_{E^*}\ell, x\rangle_{E^*,E} = \langle J_{E^*}\ell, J_E x\rangle_{E^{***},E^{**}} = \langle J_E x, \ell\rangle_{E^{**},E^*} = \langle \ell, x\rangle_{E^*,E}.$$

Hence $J_E^* J_{E^*}\ell = \ell$, so J_{E^*} is also an isomorphism.

Conversely, if E^* is reflexive, then by what we have shown E^{**} is also reflexive, and given that $E \subset E^{**}$ is a closed subspace (as E is Banach), by Exercise 7.2 we have that E is reflexive too.

Exercise 7.4.
Let E be a Banach space, $J : E \to E^{**}$ the natural embedding into its bidual, and denote by B_E and $B_{E^{**}}$ the respective closed unit balls. The goal of this exercise is to prove the following lemma of Goldstine: that $J(B_E)$ is dense in $B_{E^{**}}$ with respect to the weak-* topology (of E^{**} with respect to E^*).

(a) Show that it is enough to prove that, given $\xi \in B_{E^{**}}$, $\varepsilon > 0$, an integer N, and linearly independent forms $\ell_1, \dots, \ell_N \in E^*$,

$$\exists\, z \in B_E \quad \text{s.t.} \quad |\langle \ell_i, z\rangle - \langle \xi, \ell_i\rangle| \le \varepsilon \quad \text{for } i = 1, \dots, N. \tag{$\star$}$$

Solution: The only thing to prove is that one can reduce to the case where $\ell_1, \dots, \ell_N$ are linearly independent: we may suppose that the first $n \le N$ of them are linearly independent and the rest are linear combinations of them. Thus $(\star)$ for $i \le n$ implies $(\star)$ for $i > n$ up to replacing ε by $C\varepsilon$ for some constant C. The result follows by making ε smaller if necessary.

(b) Let $T : E \to \mathbb{R}^N$ be defined by $u \mapsto (\langle \ell_i, u\rangle)_{i=1}^N$. Show that T is surjective and that any map $\phi \in E^*$ vanishing on $\ker T$ is a linear combination of $\ell_1, \dots, \ell_N$.

Solution: If T were not surjective, its image would lie on a hyperplane of $\mathbb{R}^N$ and therefore there would exist $0 \neq (a_1, \dots, a_N) \in \mathbb{R}^N$ such that

$$\langle a_1\ell_1 + \cdots + a_N\ell_N, u\rangle = a_1\langle \ell_1, u\rangle + \cdots + a_N\langle \ell_N, u\rangle = 0 \qquad \forall u \in E,$$

contradicting the linear independence of the ℓ_i. In particular we get $u_i \in E$ such that $T(u_i) = e_i$ for $i = 1, \dots, N$ (here $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$).
Let now $\phi \in E^*$ vanish on $\ker T$. Given $u \in E$, since $u - \sum_{i=1}^N \langle \ell_i, u\rangle u_i$ lies in $\ker T$, we have that

$$\left\langle \phi - \sum_{i=1}^N \langle \phi, u_i\rangle \ell_i, u \right\rangle = \langle \phi, u\rangle - \sum_{i=1}^N \langle \ell_i, u\rangle\langle \phi, u_i\rangle = \left\langle \phi, u - \sum_{i=1}^N \langle \ell_i, u\rangle u_i \right\rangle = 0,$$

so $\phi = \sum_{i=1}^N \langle \phi, u_i\rangle \ell_i$.

(c) Show that given $\delta > 0$, one can find $y \in E$ with $\|y\| \le 1 + \delta$ such that $\langle \ell_i, y\rangle = \langle \xi, \ell_i\rangle$ for $i = 1, \dots, N$.
Hint: start with any solution of the system and improve it with the help of Exercise 7.1.

Solution: If $\langle \xi, \ell_i\rangle = 0$ for all i, then we may trivially choose $y = 0$ and we are done. Otherwise part (b) gives us some $x \in E$ which satisfies $Tx = (\langle \xi, \ell_i\rangle)_i$. Denoting $K := \ker T \subset E$ and $d := \operatorname{dist}(x, K)$, we have that $Tx \neq 0 \Rightarrow x \notin K \Rightarrow d > 0$. Exercise 7.1 now gives us a map $\phi : E \to \mathbb{R}$ with norm $\|\phi\| \le 1$ vanishing on K and such that $\langle \phi, x\rangle = d$.
By part (b), ϕ is a linear combination of the ℓ_i and hence $\langle \phi, x\rangle = \langle \xi, \phi\rangle$. Then $d = \langle \phi, x\rangle = \langle \xi, \phi\rangle \le \|\xi\|\|\phi\| \le 1$ and hence $\exists w \in K$ such that $\|x - w\| \le 1 + \delta$. The choice $y = x - w$ satisfies all the required properties.

(d) Conclude the proof of Goldstine's lemma.

Solution: Let $z = (1+\delta)^{-1}y \in B_E$. Then

$$|\langle \ell_i, z\rangle - \langle \xi, \ell_i\rangle| = \left|(1+\delta)^{-1}\langle \ell_i, y\rangle - \langle \ell_i, y\rangle\right| = \left(1-(1+\delta)^{-1}\right)|\langle \ell_i, y\rangle|$$
$$\leq C(1-(1+\delta)^{-1})(1+\delta) = C\delta \leq \varepsilon$$

if δ is chosen small enough.

Exercise 7.5.
We say that a Banach space $(E, \|\cdot\|)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in E$,

$$\|x\| \leq 1, \quad \|y\| \leq 1 \quad \text{and} \quad \|x-y\| \geq \varepsilon \quad \Longrightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

In this exercise we will prove the Milman–Pettis theorem: every uniformly convex Banach space is reflexive. To do that, fix $\xi \in E^{**}$ with $\|\xi\| = 1$ and $\varepsilon > 0$, let $\delta > 0$ be the corresponding number from the uniform convexity condition, and choose $\ell \in E^*$ such that $\|\ell\| = 1$ and $\langle \xi, \ell\rangle > 1 - \delta/2$. Moreover let $V := \{\zeta \in E^{**} : \langle \zeta, \ell\rangle > 1-\delta\}$.

(a) Show that the diameter of $V \cap J(B_E)$ is at most ε.

Solution: Let $x, y \in B_E$ such that $J(x), J(y) \in V$. Suppose that $\|J(x) - J(y)\| = \|x-y\| \geq \varepsilon$. Then the choice of δ implies that $\|x+y\| \leq 2-2\delta$ which contradicts the following inequality:

$$(1-\delta) + (1-\delta) < \langle J(x), \ell\rangle + \langle J(y), \ell\rangle = \langle J(x+y), \ell\rangle \leq \|J(x+y)\| = \|x+y\|.$$

(b) Show that $B_\varepsilon(\xi) \cap J(B_E) \neq \varnothing$.
Hint: take any $J(x) \in J(B_E) \cap V$ and show that $\|J(x) - \xi\| \leq \varepsilon$ using Exercise 7.4.

Solution: Since $\|\xi\| = 1$ and $V \ni \xi$ is a weak-* neighborhood, by Exercise 7.4 we may find $x \in B_E$ such that $J(x) \in V$. We know that the function $f : E^{**} \to \mathbb{R}$ defined by $\|J(x) - \cdot\|$ is weak-* lower semicontinuous (as its lower level sets are strongly closed and convex). Since, by part (a), $f^{-1}([0,\varepsilon])$ contains $J(B_E) \cap V$, and we have just shown that it is weak-*-closed, it must contain the weak-* closure of $J(B_E) \cap V$. Now it is enough to show that ξ is in this closure: this follows from the fact that, given a weak-* open set $W \ni \xi$, again by exercise 7.4, $J(B_E) \cap V \cap W \neq \varnothing$.

(c) Conclude the proof of the Milman–Pettis theorem.

Solution: Since we can take $\varepsilon > 0$ arbitrary in part (b), and since $J(E)$ is a closed subspace of E^{**} (as it is Banach), we obtain that $\xi \in J(E)$. As ξ was arbitrary, this shows that E is reflexive.

Exercise 7.6.
Let Ω be an open subset of $\mathbb{R}^n$.

(a) Show that $L^\infty(\Omega)$ is not separable.

Solution: Let $x_0 \in \Omega$ and $R > 0$ such that $B_R(x_0) \subset \Omega$, and consider the functions $\mathbb{1}_{B_r(x_0)} \in L^\infty(\Omega)$ for $0 < r < R$. Then for $0 < r, s < R$ distinct we have that

$$\left\| \mathbb{1}_{B_r(x_0)} - \mathbb{1}_{B_s(x_0)} \right\|_{L^\infty(\Omega)} = 1,$$

therefore these functions form an uncountable discrete subset and this is impossible in a separable metric space.

(b) Prove that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for every $1 \leq p < +\infty$, and hence that $L^p(\Omega)$ is separable for $1 \leq p < \infty$.

Solution: For the proof of the density of $C_c^\infty(\Omega)$, see Section 3.3 in the script. For the separability, we know that for each $R \in \mathbb{N}$, $C_c^0(\Omega \cap B_R)$ is separable with respect to the uniform convergence, so there exist countable dense collections $\{\varphi_{j,R}\}_{j\in\mathbb{N}}$.
Given $f \in L^p(\Omega)$, with $1 \leq p < +\infty$, and $\varepsilon > 0$, choose $R \in \mathbb{N}$ such that $\|f\|_{L^p(\Omega \setminus B_R)} < \varepsilon/3$ and then $g \in C_c^\infty(\Omega \cap B_R)$ with $\|f - g\|_{L^p(\Omega\cap B_R)} \leq \varepsilon/3$. Finally choose j such that

$$\varphi_{j,R} \in C_c^0(\Omega \cap B_R) \quad \text{satisfies} \quad \|\varphi_{j,R} - g\|_{C^0(\Omega\cap B_R)} \leq \frac{\varepsilon}{3|B_R|^{1/p}}.$$

We then have that

$$\begin{aligned}
\|f - \varphi_{j,R}\|_{L^p(\Omega)} &= \|f\|_{L^p(\Omega\setminus B_R)} + \|f - \varphi_{j,R}\|_{L^p(\Omega\cap B_R)} \\
&= \|f\|_{L^p(\Omega\setminus B_R)} + \|f - g\|_{L^p(\Omega\cap B_R)} + \|g - \varphi_{j,R}\|_{L^p(\Omega\cap B_R)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |B_R|^{1/p} \|g - \varphi_{j,R}\|_{C^0(\Omega\cap B_R)} \leq \varepsilon
\end{aligned}$$

as desired.

Exercise 7.7.

Proof the so-called Littlewood inequality in a measure space (X, μ): given $1 \leq p_0 < p_1 \leq +\infty$ and $t \in (0, 1)$, define $p_t \in [1, +\infty]$ by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1};$$

then for any $f \in L^{p_0}(X, \mu) \cap L^{p_1}(X, \mu)$ it holds that

$$f \in L^{p_t}(X, \mu) \qquad \text{with} \qquad \|f\|_{L^{p_t}} \leq \|f\|_{L^{p_0}}^{1-t} \, \|f\|_{L^{p_1}}^{t}.$$

Solution: Let $f \in L^{p_0}(X, \mu) \cap L^{p_1}(X, \mu)$ and write $|f| = |f|^{1-t}|f|^t$. Then Hölder's inequality with exponents $p_0/(1-t)$ and p_1/t, which satisfy

$$\frac{1}{p_t} = \frac{1}{p_0/(1-t)} + \frac{1}{p_1/t},$$

gives

$$\begin{aligned}
\|f\|_{L^{p_t}} &\leq \left\||f|^{1-t}\right\|_{L^{p_0/(1-t)}} \left\||f|^{t}\right\|_{L^{p_1/t}} \\
&= \left(\int_X \left(|f|^{1-t}\right)^{p_0/(1-t)} \,\mathrm{d}\mu\right)^{(1-t)/p_0} \left(\int_X \left(|f|^{t}\right)^{p_1/t} \,\mathrm{d}\mu\right)^{t/p_1} \\
&= \|f\|_{L^{p_0}}^{1-t} \, \|f\|_{L^{p_1}}^{t}.
\end{aligned}$$