Exercise 8.1.
In this exercise we will show that the range of exponents of the Hausdorff–Young inequality is sharp. To do that, consider $f(x) = e^{-x^2/2}$ and fix some parameter $a > 0$ ($a = 10$ will do). Then for $N \in \mathbb{N}$ define

$$f_N(x) := \sum_{j=1}^{N} e^{ixja} f(x - ja).$$

(a) Compute $\hat{f}_N$.

Solution: We compute directly

$$\begin{aligned}
\hat{f}_N(\xi) &= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \int_{\mathbb{R}} e^{-ix\xi} e^{ixja} f(x - ja)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \int_{\mathbb{R}} e^{-ix(\xi - ja)} f(x - ja)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \int_{\mathbb{R}} e^{-i(y+ja)(\xi - ja)} f(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} e^{-ija(\xi - ja)} \int_{\mathbb{R}} e^{-iy(\xi - ja)} f(y)\,\mathrm{d}y \\
&= \sum_{j=1}^{N} e^{-ija\xi} e^{i(ja)^2} \hat{f}(\xi - ja) = \sum_{j=1}^{N} e^{-ija\xi} e^{i(ja)^2} f(\xi - ja),
\end{aligned}$$

using that the Fourier transform of a Gaussian is also a Gaussian.

(b) Show that for all $p \in [1, \infty]$ there is a constant $c > 0$ such that $\|f_N\|_{L^p(\mathbb{R})} \geq cN^{1/p}$ and $\|\hat{f}_N\|_{L^p(\mathbb{R})} \geq cN^{1/p}$.

Solution: Observe that $e^{-x^2/2} \geq \frac{1}{2}$ for $x \in [-1, 1]$. Then it follows that, for each $k \in \{1, \ldots, N\}$ and $x \in [ka - 1, ka + 1]$,

$$f_N(x) = e^{ixka} f(x - ka) + \sum_{\substack{1 \leq j \leq N \\ j \neq k}} e^{ixja} f(x - ja) = e^{ixka} \left(f(x - ka) + \sum_{\substack{1 \leq j \leq N \\ j \neq k}} e^{ix(j-k)a} f(x - ja) \right)$$

and so, using the fact that $|x - ja| \geq |k - j|a - 1 \geq |k - j|a/2$ if $j \neq k$,

$$|f_N(x)| \geq |f(x - ka)| - \sum_{j \neq k} e^{-(x - ja)^2/2} \geq \frac{1}{2} - 2\sum_{\ell=1}^{\infty} e^{-a^2\ell^2/8} \geq \frac{1}{2} - 2\sum_{\ell=1}^{\infty} e^{-a^2\ell/8} \geq \frac{1}{4}$$

if a is large enough (for example $a = 10$). From this the case $p = \infty$ is trivial, and the case $1 \leq p < \infty$ follows from the computation

$$\int_{\mathbb{R}} |f_N(x)|^p\,\mathrm{d}x \geq \sum_{k=1}^{N} \int_{[ka-1, ka+1]} |f_N(x)|^p\,\mathrm{d}x \geq \frac{2}{4^p} N = cN.$$

The same proof works for $\hat{f}_N$, since it has the same expression up to a phase for each summand.

(c) Show that for some constant $C > 0$, $\|f_N\|_{L^1(\mathbb{R})} \leq CN$ and $\|\hat{f}_N\|_{L^1(\mathbb{R})} \leq CN$ for all N.

Solution: This follows from the triangle inequality (and similarly for $\hat{f}_N$):

$$\left\| \sum_{j=1}^{N} e^{ija\cdot} f(\cdot - ja) \right\|_{L^1(\mathbb{R})} \le \sum_{j=1}^{N} \left\| e^{ija\cdot} f(\cdot - ja) \right\|_{L^1(\mathbb{R})} = N\|f\|_{L^1(\mathbb{R})} = CN.$$

(d) Show that for some constant $C' > 0$, $\|f_N\|_{L^\infty(\mathbb{R})} \le C'$ and $\|\hat{f}_N\|_{L^\infty(\mathbb{R})} \le C'$ for all N.

Solution: This follows from a similar computation to part (b). We do it for f_N, since it's the same for $\hat{f}_N$. Given $x \in \mathbb{R}$, choose $k \in \mathbb{Z}$ such that $ka \le x < (k+1)a$. Then it is easy to see that

$$|f_N(x)| \le \sum_{j\in\mathbb{Z}} e^{-(x-ja)^2/2} \le 2\sum_{\ell=0}^{\infty} e^{-(\ell a)^2/2} \le 2\sum_{\ell=0}^{\infty} e^{-a^2\ell/2} < +\infty.$$

(e) Conclude that, if $p \in [1, \infty]$ is such that the Fourier transform is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$, then necessarily $p \le 2$.

Solution: By interpolation we have that for each $p \in [1, \infty]$,

$$\|f_N\|_{L^p(\mathbb{R})} \le C'' N^{1/p}. \tag{$\star$}$$

If the Hausdorff–Young inequality is valid for a certain p, putting ($\star$) together with part (b) we would have

$$cN^{1-1/p} = cN^{1/p'} \le \|\hat{f}_N\|_{L^{p'}} \le C\|f_N\|_{L^p} \le CN^{1/p},$$

which by letting $N \to \infty$ can only be true if $p \le 2$.

Exercise 8.2.

In this exercise we will prove the "integral Minkowski inequality": let (X, μ) and (Y, ν) be two σ-finite measure spaces[1] and let $f : X \times Y \to [0, \infty)$ be measurable with respect to the product measure. Show that for each $1 \le p < +\infty$ it holds:

$$\left(\int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y) \right)^p \mathrm{d}\mu(x) \right)^{1/p} \le \int_Y \left(\int_X f(x,y)^p\,\mathrm{d}\mu(x) \right)^{1/p} \mathrm{d}\nu(y).$$

Hint: look at what inequality you get when $(Y, \nu) = (\{1, 2\}, \#)$ and try to replicate the proof of that inequality from Measure Theory.

Solution: Let $F : X \to [0, \infty]$ be the μ-measurable function defined by $x \mapsto \int_Y f(x,y)\,\mathrm{d}\nu(y)$, and

[1] You can just take them to be measurable subsets of Euclidean space with the Lebesgue measure—we only need that the product measure is well defined and that Tonelli's theorem holds.

choose $g \in L^{p'}(X,\mu)$. Then, using Tonelli's theorem,

$$\int_X F(x)g(x)\,\mathrm{d}\mu(x) = \int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right) g(x)\,\mathrm{d}\mu(x) = \int_Y \left(\int_X f(x,y)g(x)\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y)$$
$$\leq \|g\|_{L^{p'}(X,\mu)} \int_Y \|f(\cdot,y)\|_{L^p(X,\mu)}\,\mathrm{d}\nu(y).$$

By taking g to be truncations of F^{p-1} (or by using the dual characterization of the L^p norm) this implies that $F \in L^p(X,\mu)$ with norm

$$\left\|\int_Y f(\cdot,y)\,\mathrm{d}\nu(y)\right\|_{L^p(X,\mu)} = \|F\|_{L^p(X,\mu)} \leq \int_Y \|f(\cdot,y)\|_{L^p(X,\mu)}\,\mathrm{d}\nu(y),$$

which is the desired inequality.

Exercise 8.3.
Fix $1 \leq p \leq \infty$ and suppose that $K : (0,\infty)\times(0,\infty) \to \mathbb{R}$ satisfies the following two properties:

- K is homogeneous of degree -1, that is, for $\lambda > 0$, $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$.
- it holds that $A_K := \int_0^\infty |K(1,y)|y^{-1/p}\,\mathrm{d}y < +\infty$.

We define the linear operator

$$(Tf)(x) := \int_0^\infty K(x,y)f(y)\,\mathrm{d}y;$$

show that $\|Tf\|_{L^p} \leq A_K\|f\|_{L^p}$.
Hint: write the function $(Tf)(x)$ as an integral of functions of x depending on some other parameter, and apply the integral Minkowski inequality.

Solution: We rewrite $K(x,y) = x^{-1}K(x,y/x)$ using the homogeneity condition, and use the change of variables $y = tx$:

$$(Tf)(x) = \int_0^\infty K(x,y)f(y)\,\mathrm{d}y = \int_0^\infty K\left(1,\frac{y}{x}\right) f(y)\,\frac{\mathrm{d}y}{x} = \int_0^\infty K(1,t)\,f(tx)\,\mathrm{d}t,$$

hence by Exercise 8.2,

$$\|Tf\|_{L^p} \leq \int_0^\infty \|K(1,t)\,f(t\cdot)\|_{L^p}\,\mathrm{d}t = \int_0^\infty |K(1,t)|\,\|f(t\cdot)\|_{L^p}\,\mathrm{d}t.$$

We have that

$$\|f(t\cdot)\|_{L^p}^p = \int f(tx)^p\,\mathrm{d}x = \int f(y)^p\,\frac{\mathrm{d}y}{t} = t^{-1}\|f\|_{L^p}^p,$$

so substituting above gives

$$\|Tf\|_{L^p} \leq \|f\|_{L^p}\int_0^\infty |K(1,t)|t^{-1/p}\,\mathrm{d}t = A_K\|f\|_{L^p}.$$

Exercise 8.4.

(a) Show the following version of the Hardy inequality: given a measurable function $g : (0,\infty) \to \mathbb{R}$ and two real numbers $1 \le p < \infty$ and $r > 0$,

$$\int_0^\infty \left(\int_0^x |g(y)| \,\mathrm{d}y \right)^p x^{-r-1} \,\mathrm{d}x \le \left(\frac{p}{r} \right)^p \int_0^\infty (y|g(y)|)^p y^{-r-1} \,\mathrm{d}y.$$

Hint: deduce it from the estimate of Exercise 8.3.

Solution: Choose $K(x,y) := y^{-1} \mathbb{1}_{y<x} (y/x)^{\frac{r+1}{p}}$ and define $f(y) := y|g(y)|y^{-\frac{r+1}{p}}$. Then

$$Tf(x) = \int_0^\infty K(x,y) f(y) \,\mathrm{d}y = \int_0^x y^{-1} \left(\frac{y}{x} \right)^{\frac{r+1}{p}} y|g(y)|y^{-\frac{r+1}{p}} \,\mathrm{d}y = x^{-\frac{r+1}{p}} \int_0^x |g(y)| \,\mathrm{d}y$$

The left hand side of the estimate from Exercise 8.3 is

$$\|Tf\|_{L^p}^p = \int_0^\infty \left(x^{-\frac{r+1}{p}} \int_0^x |g(y)| \,\mathrm{d}y \right)^p \mathrm{d}x = \int_0^\infty x^{-r-1} \left(\int_0^x |g(y)| \,\mathrm{d}y \right)^p \mathrm{d}x$$

while the right hand side has

$$\|f\|_{L^p}^p = \int_0^\infty (y|g(y)|)^p y^{-r-1} \,\mathrm{d}y$$

with the constant

$$A_K = \int_0^\infty K(1,t) t^{-1/p} \,\mathrm{d}t = \int_0^1 t^{-1} t^{\frac{r+1}{p}} t^{-1/p} \,\mathrm{d}t = \int_0^1 t^{\frac{r}{p}-1} \,\mathrm{d}t = \frac{p}{r}.$$

The inequality follows immediately.

(b) Obtain the following more common version of the Hardy inequality: if $u : [0,\infty) \to \mathbb{R}$ is an absolutely continuous function[2] with $u(0) = 0$, then for any $p > 1$ it holds

$$\int_0^\infty \left(\frac{|u(x)|}{x} \right)^p \mathrm{d}x \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p \,\mathrm{d}x.$$

Solution: Just take $g = u'$ in the above inequality and choose $r = p - 1 > 0$.

Exercise 8.5.

This exercise assumes familiarity with the Riesz representation theorem for measures and the Radon–Nikodym theorem.

The goal of this exercise is to give a different proof of the fact that the dual of L^p is $L^{p'}$ (where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$) using two classical theorems from Measure Theory instead of abstract Functional Analysis. For simplicity we deal with an open set $\Omega \subset \mathbb{R}^n$, which we

[2]This just means that u is the primitive of an L^1 function.

write as an increasing union of bounded open sets $\Omega_1 \subset \Omega_2 \subset \cdots$. Let $\ell : L^p(\Omega) \to \mathbb{R}$ be a linear bounded functional.

(a) Show that for each j, ℓ naturally defines a linear bounded functional on $C_c(\Omega_j)$. Therefore, by the Riesz representation theorem, we get a *signed*[3] Radon measure ν_j on Ω_j.

Solution: Given $\varphi \in C_c(\Omega_j)$, Hölder's inequality gives directly the desired estimate:

$$\langle \ell, \varphi \rangle \leq \|\ell\| \|\varphi\|_{L^p(\Omega)} = \|\ell\| \|\varphi\|_{C(\Omega_j)} |\Omega_j|^{1/p}.$$

(b) Show that for every j, $|\nu_j| \ll \mathcal{L}^n$. Hence, by the Radon–Nikodym theorem we get measurable functions $f_j \in L^1(\Omega_j)$ such that $\mathrm{d}\nu_j = f_j \mathrm{d}\mathcal{L}^n$. Show also that the functions f_j and f_{j+1} agree almost everywhere on Ω_j and hence they define a global function $f \in L^1_{\mathrm{loc}}(\Omega)$.

Solution: Suppose that $A \subset \Omega_j$ is Borel with $|A| = 0$. Then $\forall \varepsilon > 0$ there is an open set $U \subset \Omega_j$ containing A such that $|U| \leq \varepsilon$. Recall that the total variation of the representing measure is defined on open sets by

$$|\nu_j|(U) = \sup\{\langle \ell, \varphi \rangle : \varphi \in C_c(\Omega_j), |\varphi| \leq 1, \operatorname{supp} \varphi \subseteq U\}.$$

The above computation shows that

$$\langle \ell, \varphi \rangle = \|\ell\| \|\varphi\|_{C(\Omega_j)} |U|^{1/p} \leq \|\ell\| \, \varepsilon^{1/p},$$

so we deduce that $|\nu_j|(A) \leq |\nu_j|(U) \leq \|\ell\| \varepsilon^{1/p}$ and letting $\varepsilon \to 0$ follows $|\nu_j|(A) = 0$. The Radon–Nikodym theorem now provides us with said functions f_j; moreover, given $\varphi \in C_c(\Omega_j) \subset C_c(\Omega_{j+1})$,

$$\int_{\Omega_j} f_j \varphi \, \mathrm{d}\mathcal{L}^n = \int_{\Omega_j} \varphi \, \mathrm{d}\nu_j = \langle \ell, \varphi \rangle = \int_{\Omega_{j+1}} \varphi \, \mathrm{d}\nu_{j+1}, = \int_{\Omega_j} f_{j+1} \varphi \, \mathrm{d}\mathcal{L}^n$$

and by the usual approximation argument one deduces that $f_j \equiv f_{j+1}$ almost everywhere on Ω_j.

(c) Show that $f \in L^{p'}(\Omega)$ and conclude.

Solution: Fix $j \in \mathbb{N}$ and $\lambda > 0$, and consider the function $f_\lambda := \min(\lambda, |f|) \in L^\infty(\Omega_j)$, so that $g_\lambda := \operatorname{sgn}(f) f_\lambda^{p'-1} \in L^p(\Omega_j)$. Since $p < \infty$, we may choose a sequence $(\varphi_k) \subset C_c(\Omega_j)$ such that $\varphi_k \to g_\lambda$ in $L^p(\Omega_j)$ and almost everywhere. Notice that $f_\lambda^{p'} \leq |f| f_\lambda^{p'-1} = f g_\lambda$. Moreover, after truncating φ_k if necessary, we may assume that $|\varphi_k| \leq \lambda^{p'-1}$, so that $|f \varphi_k| \leq \lambda^{p'-1} |f| \in L^1(\Omega_j)$. Then the Dominated Convergence Theorem implies:

$$\begin{aligned} \int_{\Omega_j} f_\lambda^{p'} \leq \int_{\Omega_j} f_j g_\lambda &= \lim_{k \to \infty} \int_{\Omega_j} f_j \varphi_k = \lim_{k \to \infty} \langle \ell, \varphi_k \rangle \leq \lim_{k \to \infty} \|\ell\| \|\varphi_k\|_{L^p(\Omega_j)} \\ &= \|\ell\| \|g_\lambda\|_{L^p(\Omega_j)} = \|\ell\| \left(\int_{\Omega_j} f_\lambda^{p(p'-1)} \right)^{1/p} = \|\ell\| \left(\int_{\Omega_j} f_\lambda^{p'} \right)^{1/p}. \end{aligned}$$

[3] A signed Radon measure ν is just the difference of two positive (usual) Radon measures ν^+ and ν^-; this decomposition is unique if ν^+ and ν^- are mutually orthogonal, and in this case we denote the total variation measure by $|\nu| := \nu^+ + \nu^-$.

Dividing both sides we obtain that

$$\|f_\lambda\|_{L^{p'}(\Omega_j)} = \left(\int_{\Omega_j} f_\lambda^{p'}\right)^{1-1/p} \leq \|\ell\|,$$

and we get that $f \in L^p(\Omega)$ by first letting $\lambda \to \infty$ and then $j \to \infty$ with the monotone convergence theorem. The fact that f represents ℓ follows from the density of $C_c(\Omega)$ in $L^p(\Omega)$.