The first three exercises in this series are dedicated to the proof of the general Marcinkiewicz interpolation theorem (Theorem 4.3 in the script) given in Appendix B of Stein's book[1].

Exercise 9.1.
In this exercise we will prove the following inequality for a non-increasing function $h : (0,\infty) \to [0,\infty)$:

$$\left(\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{\mathrm{d}t}{t}\right)^{1/q_2} \leq A\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1},$$

where $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$ and A is a constant depending on q_1, q_2 and p.

(a) Show this first for $q_2 = \infty$ (where the left hand side is interpreted as the supremum).

Solution: For every $0 < s < \infty$ we have

$$\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t} \geq \int_{s/2}^s [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t} \geq \left[\left(\frac{s}{2}\right)^{1/p} h(s)\right]^{q_1}\int_{s/2}^s \frac{\mathrm{d}t}{t} = 2^{-1/p}\log 2\,\left[s^{1/p}h(s)\right]^{q_1}.$$

Taking the supremum over s now gives

$$\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1} \geq c\sup_{0<t<\infty}\left[t^{1/p}h(t)\right],$$

which is the desired inequality.

(b) Then show it for every $q_1 < q_2 < \infty$.

Solution: This follows from Hölder's inequality and part (a):

$$\begin{aligned}
\left(\int_0^\infty [t^{1/p}h(t)]^{q_2}\frac{\mathrm{d}t}{t}\right)^{1/q_2} &\leq \left[\sup_{0<t<\infty} t^{1/p}h(t)\right]^{1-q_1/q_2}\left[\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1}\right]^{q_1/q_2} \\
&\leq \left[A\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1}\right]^{1-q_1/q_2}\left[\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1}\right]^{q_1/q_2} \\
&= A'\left(\int_0^\infty [t^{1/p}h(t)]^{q_1}\frac{\mathrm{d}t}{t}\right)^{1/q_1}.
\end{aligned}$$

Exercise 9.2.
Prove the "second Hardy inequality": for a measurable function $f : (0,\infty) \to [0,\infty)$, and numbers $p \geq 1$ and $r > 0$,

$$\left(\int_0^\infty\left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{1/p} \leq \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{r-1}\,\mathrm{d}y\right)^{1/p}.$$

[1]Stein, Elias M. *Singular Integrals and Differentiability Properties of Functions*, Princeton: Princeton University Press, 1971. https://doi.org/10.1515/9781400883882

Hint: recall the proof of the first Hardy inequality (Exercise 8.4).

Solution: As in Exercise 8.4, we will apply Exercise 8.3 but to the function $g(y) := f(y)y^{1+(r-1)/p}$ and with the choice $K(x,y) = \mathbb{1}_{y>x}y^{-1}(x/y)^{(r-1)/p}$, which is clearly homogeneous of degree -1. Recall that we have

$$\left(\int_0^\infty |Tg(x)|^p \,\mathrm{d}x\right)^{1/p} \leq A_K \left(\int_0^\infty |g(y)|^p \,\mathrm{d}y\right)^{1/p}. \tag{1}$$

where

$$Tg(x) = \int_0^\infty K(x,y)g(y)\,\mathrm{d}y,$$

and

$$A_K = \int_0^\infty |K(1,y)|y^{-1/p}\,\mathrm{d}y = \int_0^\infty \mathbb{1}_{y>1}y^{-1-(r-1)/p}y^{-1/p}\,\mathrm{d}y = \int_1^\infty y^{-1-r/p}\,\mathrm{d}y = \frac{p}{r}.$$

The left hand side of (1) is

$$\begin{aligned}\left(\int_0^\infty |Tg(x)|^p \,\mathrm{d}x\right)^{1/p} &= \left(\int_0^\infty \left(\int_x^\infty f(y)y^{1+(r-1)/p}y^{-1-(r-1)/p}x^{(r-1)/p}\,\mathrm{d}y\right)^p \mathrm{d}x\right)^{1/p}\\ &= \left(\int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{1/p}\end{aligned}$$

and the right hand side gives

$$A_K\left(\int_0^\infty |g(y)|^p\,\mathrm{d}y\right)^{1/p} = \frac{p}{r}\left(\int_0^\infty f(y)^p y^{p(1+(r-1)/p)}\,\mathrm{d}y\right)^{1/p} = \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{r-1}\,\mathrm{d}y\right)^{1/p},$$

so the result follows.

Exercise 9.3.
The goal of this (long) exercise is to prove the general form of the Marcinkiewicz interpolation theorem. Assume we are given exponents

$$1 \leq p_0 \leq q_0 \leq \infty \quad \text{and} \quad 1 \leq p_1 \leq q_1 \leq \infty \quad \text{with} \quad p_0 < p_1 \quad \text{and} \quad q_0 \neq q_1.$$

Let T be a sub-additive operator defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ and assume that T is of weak type (p_i, q_i) for $i = 0, 1$, meaning that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}) \leq \left(\frac{A_i\|f\|_{L^{p_i}}}{\alpha}\right)^{q_i} \qquad \forall \alpha > 0$$

in case $q_i < \infty$, and in case $q_i = \infty$, that $\|Tf\|_{L^\infty} \leq A_i\|f\|_{L^{p_i}}$. The theorem then states that, given $0 < \theta < 1$ and letting

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then T is of strong type (p,q), meaning that $\|Tf\|_{L^q} \leq A\|f\|_{L^p}$ for a constant A depending on p_0, p_1, q_0, q_1 and θ.

We define the parameter σ as

$$\sigma = \frac{1/q_0 - 1/q}{1/p_0 - 1/p} = \frac{1/q - 1/q_1}{1/p - 1/p_1}.$$

For $f \in L^p(\mathbb{R}^n)$, we define its non-increasing rearrangement f^* as in Section 6.6 of the script. Then, for $t > 0$, we let

$$f^t(x) := \begin{cases} f(x) & \text{if } |f(x)| > f^*(t^\sigma), \\ 0 & \text{otherwise} \end{cases}$$

and define $f_t := f - f^t$.

(a) Check the following properties (a drawing may help!):

- $(f^t)^*(y) \leq f^*(y)$ if $0 \leq y \leq t^\sigma$;
- $(f^t)^*(y) = 0$ if $y > t^\sigma$.

Solution:

- Clearly $|f^t| \leq |f|$, so it follows that $(f^t)^* \leq f^*$.
- To show that $(f^t)^*(y) = 0$ for $y > t^\sigma$ it is enough to see that $|\{|f^t| > 0\}| < y$. But $|f^t(x)| > 0 \Rightarrow |f(x)| > f^*(t^\sigma)$ and therefore

$$|\{|f^t| > 0\}| \leq |\{|f| > f^*(t^\sigma)\}| = |\{|f^*| > f^*(t^\sigma)\}| \leq t^\sigma < y.$$

  Here we have used the equality of the distribution functions of f and f^*, and the fact that, since f^* is non-increasing, $f^*(s) \leq f^*(t^\sigma)$ for $s \geq t^\sigma$, so $|f^*(s)|$ can be bigger than $f^*(t^\sigma)$ only for $0 \leq s < t^\sigma$.

(b) Check also that

- $(f_t)^*(y) \leq f^*(t^\sigma)$ if $y \leq t^\sigma$;
- $(f_t)^*(y) \leq f^*(y)$ if $y \geq t^\sigma$.

Solution:

- We have that

$$f_t(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq f^*(t^\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

  Therefore $|f_t| \leq f^*(t^\sigma)$ and hence $(f_t)^* \leq f^*(t^\sigma)$ too.
- Again $|f_t| \leq |f|$ and therefore $(f_t)^* \leq f^*$.

(c) Verify that, if $f = f_1 + f_2$, then

$$(Tf)^*(t) \leq (Tf_1)^*(t/2) + (Tf_2)^*(t/2).$$

Solution: Let $\lambda_1 := (Tf_1)^*(t/2)$ and $\lambda_2 := (Tf_2)^*(t/2)$; then

$$|\{|Tf_1| > \lambda_1\}| \leq \frac{t}{2} \qquad \text{and} \qquad |\{|Tf_2| > \lambda_2\}| \leq \frac{t}{2}$$

because the infimum defining the non-increasing rearrangement is actually a minimum. Since $|Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)|$, it is immediate to see that

$$\{|Tf| > \lambda_1 + \lambda_2\} \subseteq \{|Tf_1| > \lambda_1\} \cup \{|Tf_2| > \lambda_2\},$$

so that

$$|\{|Tf| > \lambda_1 + \lambda_2\}| \leq |\{|Tf_1| > \lambda_1\}| + |\{|Tf_2| > \lambda_2\}| \leq \frac{t}{2} + \frac{t}{2} = t.$$

This gives that $(Tf)^*(t) \leq \lambda_1 + \lambda_2 = (Tf_1)^*(t/2) + (Tf_2)^*(t/2)$.

(d) Show that, if $f \in L^p(\mathbb{R}^n)$, $f^t \in L^{p_0}$ and $f_t \in L^{p_1}$.

Solution: On one hand, clearly $\|f^t\|_{L^{p_0}(\mathbb{R}^n)} = \|(f^t)^*\|_{L^{p_0}([0,\infty))}$ and, thanks to (a) and Hölder's inequality,

$$\begin{aligned}
\|(f^t)^*\|_{L^{p_0}([0,\infty))}^{p_0} &= \int_0^\infty (f^t)^*(y)^{p_0}\,\mathrm{d}y \leq \int_0^{t^\sigma} f^*(y)^{p_0}\,\mathrm{d}y \\
&\leq \left(\int_0^{t^\sigma} f^*(y)^p\,\mathrm{d}y\right)^{p_0/p} \left(\int_0^{t^\sigma} \mathrm{d}y\right)^{1-p_0/p} \\
&\leq (t^\sigma)^{1-p_0/p}\,\|f^*\|_{L^p}^{p_0} = (t^\sigma)^{1-p_0/p}\,\|f\|_{L^p}^{p_0} < +\infty.
\end{aligned}$$

On the other hand, $|f_t| \leq f^*(t^\sigma)$ everywhere, so $f_t \in L^\infty$ and thus $f_t \in L^{p_1}$.

(e) Prove the estimate

$$(Tf)^*(t) \leq A_0(2/t)^{1/q_0}\|f^t\|_{L^{p_0}} + A_1(2/t)^{1/q_1}\|f_t\|_{L^{p_1}}. \tag{2}$$

Solution: For any $\lambda < (Tf^t)^*(t/2)$, by the definition of the non-increasing rearrangement and the weak type (p_0, q_0), we have that

$$\frac{t}{2} < |\{|Tf^t| > \lambda\}| \leq \left(\frac{A_0\|f^t\|_{L^{p_0}}}{\lambda}\right)^{q_0},$$

which gives $\lambda < A_0(2/t)^{1/q_0}\|f^t\|_{L^{p_0}}$ and then, after letting $\lambda \nearrow (Tf^t)^*(t/2)$,

$$(Tf^t)^*(t/2) \leq A_0 \left(\frac{2}{t}\right)^{1/q_0} \|f^t\|_{L^{p_0}}.$$

Analogously we have

$$(Tf_t)^*(t/2) \leq A_1 \left(\frac{2}{t}\right)^{1/q_1} \|f_t\|_{L^{p_1}}.$$

Adding up these two expressions and using part (c) with $f = f^t + f_t$ yields the estimate.

(f) Using Exercise 9.1, show that

$$\|Tf\|_{L^q} \leq C\left(\int_0^\infty (t^{1/q}(Tf)^*(t))^p \frac{\mathrm{d}t}{t}\right)^{1/p} \tag{3}$$

for a constant $C > 0$.

Solution: We apply Exercise 9.1 with the function $h(t) = (Tf)^*(t)$ and the exponents q and $p \leq q$:

$$\left(\int_0^\infty h(t)^q \,\mathrm{d}t\right)^{1/q} = \left(\int_0^\infty [t^{1/q}h(t)]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \leq A\left(\int_0^\infty [t^{1/q}h(t)]^p \frac{\mathrm{d}t}{t}\right)^{1/p}.$$

The result follows by recalling that $\|Tf\|_{L^q(\mathbb{R}^n)} = \|(Tf)^*\|_{L^q([0,\infty))}$.

(g) Show that, in order to prove the theorem, it is enough to show the following two estimates:

$$\left(\int_0^\infty [t^{1/q-1/q_0}\|f^t\|_{L^{p_0}}]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \leq C\|f\|_{L^p} \tag{4}$$

and

$$\left(\int_0^\infty [t^{1/q-1/q_1}\|f_t\|_{L^{p_1}}]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \leq C\|f\|_{L^p}. \tag{5}$$

Solution: Assuming (4) and (5), we have that

$$\begin{aligned}
\|Tf\|_{L^q(\mathbb{R}^n)} &\overset{\text{by (3)}}{\leq} C\left(\int_0^\infty (t^{1/q}(Tf)^*(t))^p \frac{\mathrm{d}t}{t}\right)^{1/p} \\
&\overset{\text{by (2)}}{\leq} C\left(\int_0^\infty \left[t^{1/q}\left(A_0(2/t)^{1/q_0}\|f^t\|_{L^{p_0}} + A_1(2/t)^{1/q_1}\|f_t\|_{L^{p_1}}\right)\right]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \\
&\overset{\text{by Minkowski}}{\leq} C\left(\int_0^\infty \left[t^{1/q-1/q_0}\|f^t\|_{L^{p_0}}\right]^p \frac{\mathrm{d}t}{t}\right)^{1/p} + C\left(\int_0^\infty \left[t^{1/q-1/q_1}\|f_t\|_{L^{p_1}}\right]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \\
&\overset{\text{by (4) \& (5)}}{\leq} C\|f\|_{L^p}.
\end{aligned}$$

(h) Show, using again Exercise 9.1, that

$$\|f^t\|_{L^{p_0}} \leq C\int_0^{t^\sigma} y^{1/p_0} f^*(y) \frac{\mathrm{d}y}{y}. \tag{6}$$

Solution: We use Exercise 9.1 with the exponents p_0 and $1 \leq p_0$:

$$\|f^t\|_{L^{p_0}} = \|(f^t)^*\|_{L^{p_0}} = \left(\int_0^\infty \left[y^{1/p_0}(f^t)^*(y)\right]^{p_0} \frac{\mathrm{d}y}{y}\right)^{1/p_0} \leq C\int_0^\infty \left[y^{1/p_0}(f^t)^*(y)\right] \frac{\mathrm{d}y}{y}$$

together with the properties from part (a):

$$\int_0^\infty y^{1/p_0}(f^t)^*(y)\,\frac{\mathrm{d}y}{y} = \int_0^{t^\sigma} y^{1/p_0}(f^t)^*(y)\,\frac{\mathrm{d}y}{y} \leq \int_0^{t^\sigma} y^{1/p_0} f^*(y)\,\frac{\mathrm{d}y}{y}.$$

(i) Prove (4) by using (6) and the Hardy inequality (Exercise 8.4).

Solution: We plug (6) into the integral of (4) and use the change of variables $x = t^\sigma$:

$$\begin{aligned}\int_0^\infty [t^{1/q-1/q_0}\|f^t\|_{L^{p_0}}]^p \frac{\mathrm{d}t}{t} &\leq C\int_0^\infty \left[t^{1/q-1/q_0}\int_0^{t^\sigma} y^{1/p_0} f^*(y)\frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}t}{t}\\ &= \frac{C}{|\sigma|}\int_0^\infty \left[x^{\frac{1/q-1/q_0}{\sigma}}\int_0^x y^{1/p_0} f^*(y)\frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}x}{x}\\ &= C'\int_0^\infty \left[x^{1/p-1/p_0}\int_0^x y^{1/p_0} f^*(y)\frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}x}{x}.\end{aligned}$$

Now thanks to the first Hardy inequality with $r = p/p_0 - 1 > 0$ applied to $y^{1/p_0}f^*(y)/y$ we have

$$\begin{aligned}\int_0^\infty [t^{1/q-1/q_0}\|f^t\|_{L^{p_0}}]^p \frac{\mathrm{d}t}{t} &\leq C'\int_0^\infty x^{1-p/p_0}\left[\int_0^x y^{1/p_0} f^*(y)\frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}x}{x}\\ &\leq C''\int_0^\infty (y^{1/p_0}f^*(y))^p y^{-p/p_0}\,\mathrm{d}y\\ &\leq C''\int_0^\infty f^*(y)^p\,\mathrm{d}y = C''\|f^*\|^p_{L^p([0,\infty))} = C''\|f\|^p_{L^p(\mathbb{R}^n)}.\end{aligned}$$

(j) Prove (5) by a similar argument: first find an analog of (6) and then conclude by the second Hardy inequality (Exercise 9.2).

Solution: We follow the steps from before: first we apply Exercise 9.1 with exponents p_1 and $1 \leq p_1$ to the non-increasing function $(f_t)^*$:

$$\|f_t\|_{L^{p_1}(\mathbb{R}^n)} = \|(f_t)^*\|_{L^{p_1}([0,\infty))} = \left(\int_0^\infty \left[y^{1/p_1}(f_t)^*(y)\right]^{p_1}\,\frac{\mathrm{d}y}{y}\right)^{1/p_1} \leq C\int_0^\infty \left[y^{1/p_1}(f_t)^*(y)\right]\,\frac{\mathrm{d}y}{y}.$$

Now we use part (b), which allows us to split this integral:

$$\int_0^\infty y^{1/p_1}(f_t)^*(y)\,\frac{\mathrm{d}y}{y} = \int_0^{t^\sigma} y^{1/p_1} f^*(t^\sigma)\,\frac{\mathrm{d}y}{y} + \int_{t^\sigma}^\infty y^{1/p_1} f^*(y)\,\frac{\mathrm{d}y}{y}.$$

Notice that the first summand is just

$$\int_0^{t^\sigma} y^{1/p_1} f^*(t^\sigma)\,\frac{\mathrm{d}y}{y} = p_1 f^*(t^\sigma)t^{\sigma/p_1}.$$

Thus the Minkowski inequality gives

$$\begin{aligned}\left(\int_0^\infty [t^{1/q-1/q_1}\|f_t\|_{L^{p_1}}]^p\frac{\mathrm{d}t}{t}\right)^{1/p} &\leq \left(\int_0^\infty \left[t^{1/q-1/q_1}p_1 f^*(t^\sigma)t^{\sigma/p_1}\right]^p\,\frac{\mathrm{d}t}{t}\right)^{1/p}\\ &\quad + \left(\int_0^\infty \left[t^{1/q-1/q_1}\int_{t^\sigma}^\infty y^{1/p_1} f^*(y)\,\frac{\mathrm{d}y}{y}\right]^p\,\frac{\mathrm{d}t}{t}\right)^{1/p}.\end{aligned}$$

For the first term, using the change of variable $x = t^\sigma$,

$$\int_0^\infty \left[t^{1/q-1/q_1}p_1 f^*(t^\sigma)t^{\sigma/p_1}\right]^p\,\frac{\mathrm{d}t}{t} = \frac{1}{|\sigma|}\int_0^\infty \left[x^{1/p-1/p_1}p_1 f^*(x)x^{1/p_1}\right]^p\,\frac{\mathrm{d}x}{x} = C\int_0^\infty f^*(x)^p\mathrm{d}x,$$

and for the second term, using the same change of variable in addition to Exercise 9.2 with $r = 1 - p/p_1 > 0$ and the function $y^{1/p_1} f^*(y)/y$,

$$\begin{aligned}
\int_0^\infty \left[t^{1/q-1/q_1} \int_{t^\sigma}^\infty y^{1/p_1} f^*(y) \frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}t}{t} &= \frac{1}{|\sigma|} \int_0^\infty \left[x^{1/p-1/p_1} \int_x^\infty y^{1/p_1} f^*(y) \frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}x}{x} \\
&= \frac{1}{|\sigma|} \int_0^\infty x^{1-p/p_1} \left[\int_x^\infty y^{1/p_1} f^*(y) \frac{\mathrm{d}y}{y}\right]^p \frac{\mathrm{d}x}{x} \\
&\leq C \int_0^\infty (y^{1/p_1} f^*(y))^p y^{1-p/p_1-1} \,\mathrm{d}y = C \int_0^\infty f^*(y)^p \,\mathrm{d}y.
\end{aligned}$$

Putting these two estimates together gives (5).

Exercise 9.4.
Prove the following maximal function estimate for functions $f \in L \log L(\mathbb{R}^n)$: for any measurable $A \subset \mathbb{R}^n$ with finite measure,

$$\int_A |Mf|(y) \,\mathrm{d}y \leq C \int_{\mathbb{R}^n} |f|(y) \log\left(e + \mathcal{L}^n(A) \frac{|f|(y)|}{\|f\|_{L^1(\mathbb{R}^n)}}\right) \mathrm{d}y,$$

where C is a constant only depending on n. Here $L \log L$ is the space of functions $f \in L^1(\mathbb{R}^n)$ for which the right hand side is finite.

Hint: express the left hand side as an integral of the distribution function of $|Mf|$ and use inequality (5.9) from the script for large enough α (how large?).

Solution: See Theorem 5.8 in the script.