**Exercise 10.1.**
Let $1 \le p < q \le \infty$ and $f \in L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$, where $(X,\mu)$ is a $\sigma$-finite measure space (e.g. a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure). Show that for every $r$ such that $p < r < q$, $f \in L^r(X,\mu)$ with

$$\|f\|_{L^r} \le \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{1-\theta} \|f\|_{L^{q,\infty}}^{\theta},$$

where $\theta \in (0,1)$ is defined by

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

**Solution:** We prove the statement for $q < \infty$ first. Notice that

$$|\{|f| > \lambda\}| \le \min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\lambda^q}\right).$$

Let us try to estimate the $L^r$ norm of $f$:

$$\begin{aligned}
\|f\|_{L^r}^r = r\int_0^\infty \lambda^{r-1}|\{|f|>\lambda\}|\,\mathrm{d}\lambda &\le r\int_0^\infty \lambda^{r-1}\min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\lambda^q}\right)\mathrm{d}\lambda \\
&\le r\int_0^B \lambda^{r-1}\frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p}\,\mathrm{d}\lambda + r\int_B^\infty \lambda^{r-1}\frac{\|f\|_{L^{q,\infty}}^q}{\lambda^q}\,\mathrm{d}\lambda.
\end{aligned}$$

Here $B \in (0,\infty)$ is a constant to be determined. We have

$$r\int_0^B \lambda^{r-1}\frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p}\,\mathrm{d}\lambda = r\|f\|_{L^{p,\infty}}^p \int_0^B \lambda^{r-1-p}\,\mathrm{d}\lambda = \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p B^{r-p}$$

and

$$r\int_B^\infty \lambda^{r-1}\frac{\|f\|_{L^{q,\infty}}^q}{\lambda^q}\,\mathrm{d}\lambda = r\|f\|_{L^{q,\infty}}^q \int_B^\infty \lambda^{r-1-q}\,\mathrm{d}\lambda = \frac{r}{q-r}\|f\|_{L^{q,\infty}}^q B^{r-q}.$$

It is natural (and optimal, up to a constant) to choose $B$ such that these two terms are equal. This gives

$$\|f\|_{L^{p,\infty}}^p B^{r-p} = \|f\|_{L^{q,\infty}}^q B^{r-q} \quad \Longrightarrow \quad B = \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{1}{q-p}}$$

and, substituting above, we get

$$\begin{aligned}
\|f\|_{L^r}^r &\le \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{r-p}{q-p}} + \frac{r}{q-r}\|f\|_{L^{q,\infty}}^q \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{r-q}{q-p}} \\
&= \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{r-p}{q-p}} + \frac{r}{q-r}\|f\|_{L^{q,\infty}}^q \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{r-q}{q-p}}.
\end{aligned}$$

Now note that

$$q\frac{r-p}{q-p} = q\left(1+\frac{r-q}{q-p}\right) = r\theta \qquad \text{and} \qquad p\frac{q-r}{q-p} = p\left(1-\frac{r-p}{q-p}\right) = r(1-\theta),$$

so the above simplifies to

$$\begin{aligned}\|f\|_{L^r}^r &\leq \frac{r}{r-p}\|f\|_{L^{p,\infty}}^{r(1-\theta)}\|f\|_{L^{q,\infty}}^{r\theta} + \frac{r}{q-r}\|f\|_{L^{q,\infty}}^{r\theta}\|f\|_{L^{p,\infty}}^{r(1-\theta)} \\ &\leq \left(\frac{r}{r-p}+\frac{r}{q-r}\right)\left(\|f\|_{L^{q,\infty}}^{\theta}\|f\|_{L^{p,\infty}}^{1-\theta}\right)^r.\end{aligned}$$

The case $q=\infty$ is easier: denoting $A := \|f\|_{L^\infty}$, we have

$$\begin{aligned}\|f\|_{L^r}^r = r\int_0^\infty \lambda^{r-1}|\{|f|>\lambda\}|\,\mathrm{d}\lambda &= r\int_0^A \lambda^{r-1}|\{|f|>\lambda\}|\,\mathrm{d}\lambda \\ &\leq r\int_0^A \lambda^{r-1}\frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p}\,\mathrm{d}\lambda = \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p A^{r-p},\end{aligned}$$

therefore

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p}\right)^{1/r}\|f\|_{L^{p,\infty}}^{p/r}A^{1-p/r} = \left(\frac{r}{r-p}\right)^{1/r}\|f\|_{L^{p,\infty}}^{1-\theta}\|f\|_{L^\infty}^{\theta}.$$

**Exercise 10.2.**
Let $f_1,\dots,f_N \in L^{p,\infty}(X,\mu)$ for $1 \leq p < \infty$. Show that

$$\left\|\sum_{j=1}^N f_j\right\|_{L^{p,\infty}} \leq N\sum_{j=1}^N \|f_j\|_{L^{p,\infty}}.$$

**Solution:** For each $\lambda > 0$, one can easily see (by contradiction) that

$$\left|\sum_{j=1}^N f_j\right| > \lambda \Longrightarrow |f_j| > \lambda/N \text{ for at least one } j.$$

Therefore

$$\begin{aligned}\lambda\left|\left\{\left|\sum_{j=1}^N f_j\right|>\lambda\right\}\right|^{1/p} &\leq \lambda\left|\bigcup_{j=1}^N\{|f_j|>\lambda/N\}\right|^{1/p} \leq \lambda\left(\sum_{j=1}^N|\{|f_j|>\lambda/N\}|\right)^{1/p} \\ &\leq N\frac{\lambda}{N}\left(\sum_{j=1}^N|\{|f_j|>\lambda/N\}|^{1/p}\right) \leq N\sum_{j=1}^N\|f_j\|_{L^{p,\infty}},\end{aligned}$$

and we are done by taking the supremum over $\lambda > 0$. Here we have used the inequality $(a_1 + \cdots + a_N)^\alpha \le a_1^\alpha + \cdots + a_N^\alpha$, which holds whenever $a_j \ge 0$ and $0 < \alpha \le 1$. This can be proved by induction iterating on the inequality $a^\alpha + b^\alpha \ge (a+b)^\alpha$. To show this, we compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(a+t)^\alpha - t^\alpha = \alpha\left((a+t)^{\alpha-1} - t^{\alpha-1}\right) \le 0,$$

using the fact that the function $x \mapsto x^{\alpha-1}$ is nonincreasing for $\alpha \le 1$. Therefore

$$(a+b)^\alpha - b^\alpha \le (a+0)^\alpha - 0^\alpha = a^\alpha,$$

which is the desired inequality.

**Exercise 10.3.**
Let $0 < p < 1$. Prove that $L^p(X,\mu)$ is a complete quasi-normed space, i.e. that $\|f\|_{L^p} := (\int_X |f|^p \,\mathrm{d}\mu)^{1/p}$ defines a quasi-norm and every quasi-norm Cauchy sequence is quasi-norm convergent.

**Hint:** recall the proof of the corresponding theorem for $p \ge 1$.

**Solution:** First observe that $(a+b)^p \le a^p + b^p$ whenever $0 < p < 1$ (see the end of the solution of Exercise 10.2) and, by Hölder's inequality,

$$a^p + b^p = 1 \cdot a^p + 1 \cdot b^p \le (1+1)^{1-p}\left((a^p)^{1/p} + (b^p)^{1/p}\right)^p = 2^{1-p}\,(a+b)^p\,,$$

which implies that $(A+B)^{1/p} \le 2^{1/p-1}\left(A^{1/p} + B^{1/p}\right)$. Therefore

$$\left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{1/p} \le \left(\int_X |f|^p \,\mathrm{d}\mu + \int_X |g|^p \,\mathrm{d}\mu\right)^{1/p} \le K_p\left[\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{1/p}\right].$$

with $K_p = 2^{1/p-1} \in (1,\infty)$. This shows that $\|\cdot\|_{L^p}$ is a quasi-norm.

We claim the following inequality:

$$\|g_1 + \cdots + g_N\|_{L^p} \le \sum_{j=1}^{N} K_p^j \|g_j\|_{L^p}.$$

This is clear for $N = 1$ and follows easily by induction for larger $N$.

Now let $(f_j)$ be a Cauchy sequence in $L^p(X,\mu)$ for this quasi-norm, and let $(f_{j_k})$ be a subsequence such that

$$\|f_{j_{k+1}} - f_{j_k}\|_{L^p} \le (2K_p)^{-k}.$$

Consider the partial sums

$$g_k := \sum_{l=1}^{k-1} |f_{j_{l+1}} - f_{j_l}| \xrightarrow{k\to\infty} \sum_{l=1}^{\infty} |f_{j_{l+1}} - f_{j_l}| =: g$$

so that

$$\|g_k\|_{L^p} \leq \sum_{l=1}^{k-1} K_p^l \|f_{j_{l+1}} - f_{j_l}\|_{L^p} \leq \sum_{l=1}^{k-1} K_p^l (2K_p)^{-l} \leq 1$$

and thus, by the monotone convergence theorem,

$$\int_X g^p \, \mathrm{d}\mu = \lim_{k\to\infty} \int_X g_k^p \, \mathrm{d}\mu = \lim_{k\to\infty} \|g_k\|_{L^p}^p \leq 1.$$

This shows that the series $\sum_{l=1}^{\infty}(f_{j_{l+1}} - f_{j_l})$ converges absolutely, and hence converges, almost everywhere. Let us define

$$f(x) := f_{j_1}(x) + \sum_{l=1}^{\infty} f_{j_{l+1}}(x) - f_{j_l}(x)$$

for $\mu$-a.e. $x \in X$. Clearly $|f| \leq |f_{j_1}| + g$ is the sum of two functions in $L^p$, so $f \in L^p$. Moreover,

$$|f - f_{j_k}|^p = \left| \sum_{l=k}^{\infty} f_{j_{l+1}} - f_{j_l} \right|^p \leq \left( \sum_{l=k}^{\infty} |f_{j_{l+1}} - f_{j_l}| \right)^p \leq g^p \in L^1,$$

so we may apply the Dominated Convergence Theorem and pass to the limit:

$$\lim_{k\to\infty} \|f - f_{j_k}\|_{L^p}^p = \lim_{k\to\infty} \int_X |f - f_{j_k}|^p \, \mathrm{d}\mu = \int_X 0 \, \mathrm{d}\mu = 0.$$

Finally, to prove the convergence of the whole sequence, let $\varepsilon > 0$ and choose $i_0$ such that for every $i, j \geq i_0$, $\|f_i - f_j\|_{L^p} < \varepsilon/(2K_p)$. Choosing also $k$ such that $j_k \geq i_0$ so that $\|f_{j_k} - f\|_{L^p} \leq \varepsilon/(2K_p)$ we conclude.

**Exercise 10.4.**
Consider the following $N!$ functions $f_\sigma : \mathbb{R} \to \mathbb{R}$, for $\sigma \in \mathcal{S}_N$, the group of permutations of $\{1, 2, \ldots, N\}$:

$$f_\sigma := \sum_{j=1}^{N} \frac{N}{\sigma(j)} \chi_{\left[\frac{j-1}{N}, \frac{j}{N}\right)}.$$

(a) Show that $\|f_\sigma\|_{L^{1,\infty}} = 1$.

**Solution:** It is clear that

$$\lambda |\{f_\sigma > \lambda\}| \tag{$\star$}$$

is zero for $\lambda \geq 1$, and is equal to $\lambda j/N$ when $j$ is the largest integer $\leq N$ such that $\lambda < N/j$. Therefore ($\star$) is maximized when $\lambda$ approaches $N/j$, giving for each $j$ the value $\frac{N}{j} \cdot \frac{j}{N} = 1$. Thus $\|f_\sigma\|_{L^{1,\infty}} = 1$.

(b) Show that

$$\left\| \sum_{\sigma \in \mathcal{S}_n} f_\sigma \right\|_{L^{1,\infty}} = N! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} \right).$$

**Solution:** Clearly, for each $x \in [0,1)$ and for each $j \in \{1, \dots, N\}$, $f(x) = N/j$ for exactly $N!/N$ choices of $\sigma$. Thus

$$\sum_{\sigma \in \mathcal{S}_n} f_\sigma(x) = \frac{N!}{N} \left( \frac{N}{1} + \cdots + \frac{N}{N} \right) = N! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} \right)$$

and therefore the sum is zero outside of $[0,1)$ and equal to this constant in $[0,1)$. It is immediate to check then that the $L^{1,\infty}$ quasi-norm has the claimed value.

(c) Conclude that $L^{1,\infty}(\mathbb{R})$ is not normable, i.e. that there is no norm $\|\cdot\|$ on $L^{1,\infty}(\mathbb{R})$ such that, for a constant $C \geq 1$, $C^{-1}\|f\| \leq \|f\|_{L^{1,\infty}} \leq C\|f\|$ holds for every $f \in L^{1,\infty}(\mathbb{R})$.

**Solution:** Suppose that such a norm $\|\cdot\|$ and constant $C$ exist. Then we would have

$$\begin{aligned} N! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} \right) = \left\| \sum_{\sigma \in \mathcal{S}_n} f_\sigma \right\|_{L^{1,\infty}} &\leq C \left\| \sum_{\sigma \in \mathcal{S}_n} f_\sigma \right\| \leq C \sum_{\sigma \in \mathcal{S}_n} \|f_\sigma\| \\ &\leq C \sum_{\sigma \in \mathcal{S}_n} C \, \|f_\sigma\|_{L^{1,\infty}} = C^2 \sum_{\sigma \in \mathcal{S}_n} 1 = N! C^2. \end{aligned}$$

Dividing by $N!$ and letting $N \to \infty$ gives a contradiction.

## Exercise 10.5.

For a measurable function $g : \mathbb{R}^n \to [0, \infty)$, denote by $g^* : [0, \infty) \to [0, \infty)$ its decreasing rearrangement.

(a) Prove that for every measurable set $A \subset \mathbb{R}^n$,

$$\int_A g(x) \, \mathrm{d}x \leq \int_0^{|A|} g^*(t) \, \mathrm{d}t.$$

**Solution:** We obviously have that $|\{g|_A > \lambda\}| \leq |\{g > \lambda\}|$, and $|\{g|_A > \lambda\}| \leq |A|$, which implies that $(g|_A)^*(t) \leq g^*(t)$ for every $t > 0$ and $(g|_A)^*(t) = 0$ for every $t \geq |A|$, respectively. Hence

$$\int_A g(x) \, \mathrm{d}x = \int_0^\infty (g|_A)^*(t) \, \mathrm{d}t = \int_0^{|A|} (g|_A)^*(t) \, \mathrm{d}t \leq \int_0^{|A|} g^*(t) \, \mathrm{d}t.$$

(b) Show the *Hardy–Littlewood inequality*: for any measurable functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\int_{\mathbb{R}^n} f(x) g(x) \, \mathrm{d}x \leq \int_0^\infty f^*(t) g^*(t) \, \mathrm{d}t.$$

**Solution:** Using part (a), by Tonelli's theorem we have that

$$\int_{\mathbb{R}^n} f(x) g(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \int_0^\infty \mathbb{1}_{f(x) > \lambda} \mathrm{d}\lambda \, g(x) \, \mathrm{d}x = \int_0^\infty \int_{\{f > \lambda\}} g(x) \, \mathrm{d}x \, \mathrm{d}\lambda \leq \int_0^\infty \int_0^{|\{f > \lambda\}|} g^*(t) \, \mathrm{d}t \, \mathrm{d}\lambda.$$

Now observe that $t < |\{f > \lambda\}|$ if and only if $f^*(t) > \lambda$ (actually we only need the implication to the right, which is immediate to verify). Thus, using again Tonelli, we may rewrite this integral as

$$\int_0^\infty \int_0^{|\{f>\lambda\}|} g^*(t)\,\mathrm{d}t\,\mathrm{d}\lambda = \int_0^\infty \int_0^{f^*(t)} g^*(t)\,\mathrm{d}\lambda\,\mathrm{d}t = \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t.$$