**Exercise 12.1. ★**

Let $\Omega \subset \mathbb{R}^n$ be an open set. In this exercise we will show that the dual of $L^{p,q}(\Omega)$ for $1 < p < \infty$ and $1 < q < \infty$ is $L^{p',q'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

(a) First show that any $h \in L^{p',q'}(\Omega)$ induces a bounded linear functional

$$f \in L^{p,q}(\Omega) \mapsto \int_\Omega fh \,\mathrm{d}x \in \mathbb{R}.$$

**Solution:** By the Hardy–Littlewood and Hölder inequalities we have

$$\begin{aligned}\int_\Omega fh \,\mathrm{d}x &\leq \int_0^\infty f^*(t)h^*(t)\,\mathrm{d}t = \int_0^\infty t^{1/p} f^*(t) t^{1/p'} h^*(t) \,\frac{\mathrm{d}t}{t} \\ &\leq \left(\int_0^\infty t^{q/p} f^*(t)^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} \left(\int_0^\infty t^{q'/p'} h^*(t)^{q'} \,\frac{\mathrm{d}t}{t}\right)^{1/q'} = \|h\|_{L^{p',q'}} \|f\|_{L^{p,q}}.\end{aligned}$$

(b) Arguing as in Exercise 11.3, given $T : L^{p,q}(\Omega) \to \mathbb{R}$ linear bounded, we obtain a function $h \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$T(\psi) = \int_\Omega \psi h \,\mathrm{d}x$$

whenever $\psi \in C_c(\Omega)$. Denote $h_{k,N} := h \mathbb{1}_{\Omega_k} \mathbb{1}_{|h| \leq N}$. Show that for every $\varphi \in L^{p,q}((0,\infty))$,

$$\int_0^\infty \varphi(t) h^*_{k,N}(t)\,\mathrm{d}t \leq \|T\|\,\|\varphi\|_{L^{p,q}}.$$

**Hint:** use Exercise 11.2.

**Solution:** Let $f : \Omega \to \mathbb{R}$ be a function such that $f^* = \varphi$ (hence $f \in L^{p,q}(\Omega)$) and

$$\int_0^\infty \varphi(t) h^*_{k,N}(t)\,\mathrm{d}t = \int_\Omega f h_{k,N} \,\mathrm{d}x = \int_{\Omega_k} hf \mathbb{1}_{|h| \leq N} \,\mathrm{d}x. \tag{1}$$

We can find such a function by Exercise 11.2. Now, for $M \in \mathbb{N}$, let $f_M := f \mathbb{1}_{|f| \leq M} \mathbb{1}_{|h| \leq N} \mathbb{1}_{\Omega_k}$, which is in $L^\infty \cap L^1$, and choose a sequence $\psi_j \in C_c(\Omega_{k+1})$ such that $|\psi_j| \leq 2M$ and $\psi_j \to f_M$ in $L^r(\Omega_{k+1})$ for some $p < r < \infty$ and in particular also in $L^{p,q}(\Omega)$. We have

$$\lim_{j\to\infty} T(\psi_j) = T(f_M) \leq \|T\|\,\|f_M\|_{L^{p,q}} \leq \|T\|\,\|f\|_{L^{p,q}}$$

and also, by Dominated Convergence (as $|h\psi_j| \leq 2M|h| \in L^1(\Omega_{k+1})$)

$$\lim_{j\to\infty} T(\psi_j) = \lim_{j\to\infty} \int_{\Omega_{k+1}} h\psi_j = \int_\Omega hf_M = \int_{\Omega_k} hf \mathbb{1}_{|h| \leq N} \mathbb{1}_{|f| \leq M}.$$

Now let $M \to \infty$ and use Dominated Convergence again (with $|hf \mathbb{1}_{|h| \leq N} \mathbb{1}_{|f| \leq M}| \leq N|f| \in L^{p,q}(\Omega_k) \subseteq L^1(\Omega_k)$):

$$\int_{\Omega_k} hf \mathbb{1}_{|h| \leq N} = \lim_{M\to\infty} \int_{\Omega_k} hf \mathbb{1}_{|h| \leq N} \mathbb{1}_{|f| \leq M} \leq \|T\|\,\|f\|_{L^{p,q}(\Omega)} = \|T\|\,\|\varphi\|_{L^{p,q}((0,\infty))}.$$

This together with (1) shows the desired inequality.

(c) Choose now

$$\varphi(t) := \int_{t/2}^{\infty} s^{\frac{q'}{p'}-1} h_{k,N}^*(s)^{q'-1} \frac{\mathrm{d}s}{s}.$$

Prove that

$$\|\varphi\|_{L^{p,q}} \leq C(p,q)\|h_{k,N}\|_{L^{p',q'}}^{q'/q}.$$

**Solution:** Using the second Hardy inequality (Exercise 9.2), we have that

$$\begin{aligned}
\|\varphi\|_{L^{p,q}} &= \left(\int_0^{\infty} t^{\frac{q}{p}} \left[\int_{t/2}^{\infty} s^{\frac{q'}{p'}-1} h_{k,N}^*(s)^{q'-1} \frac{\mathrm{d}s}{s}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= c\left(\int_0^{\infty} t^{\frac{q}{p}} \left[\int_{t}^{\infty} s^{\frac{q'}{p'}-1} h_{k,N}^*(s)^{q'-1} \frac{\mathrm{d}s}{s}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&\leq C\left(\int_0^{\infty} t^{\frac{q}{p}} \left(t^{\frac{q'}{p'}-1} h_{k,N}^*(t)^{q'-1}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= C\left(\int_0^{\infty} t^{\frac{q}{p}+q\left(\frac{q'}{p'}-1\right)} h_{k,N}^*(t)^{q'} \frac{\mathrm{d}t}{t}\right)^{1/q}.
\end{aligned}$$

A computation gives

$$\frac{q}{p} + q\left(\frac{q'}{p'}-1\right) = \frac{q}{p} + q\frac{q(p-1)-p(q-1)}{p(q-1)} = \frac{q(q-1)+q(p-q)}{p(q-1)} = \frac{q(p-1)}{p(q-1)} = \frac{q'}{p'},$$

so we get

$$\|\varphi\|_{L^{p,q}} \leq C\left(\int_0^{\infty} t^{\frac{q'}{p'}} h_{k,N}^*(t)^{q'} \frac{\mathrm{d}t}{t}\right)^{1/q} = C\|h_{k,N}\|_{L^{p',q'}}^{q'/q}.$$

(d) Show that

$$\|h_{k,N}^*\|_{L^{p',q'}}^{q'} \leq C(p,q)\int_0^{\infty} \varphi(t) h_{k,N}^*(t)\,\mathrm{d}t.$$

**Solution:** We estimate

$$\begin{aligned}
\int_0^{\infty} \varphi(t) h_{k,N}^*(t)\,\mathrm{d}t &\geq \int_0^{\infty} h_{k,N}^*(t) \int_{t/2}^{t} s^{\frac{q'}{p'}-1} h_{k,N}^*(s)^{q'-1} \frac{\mathrm{d}s}{s}\,\mathrm{d}t \\
&\geq c\int_0^{\infty} h_{k,N}^*(t)^{q'} \int_{t/2}^{t} s^{\frac{q'}{p'}-1} \frac{\mathrm{d}s}{s}\,\mathrm{d}t \\
&= c\int_0^{\infty} h_{k,N}^*(t)^{q'} t^{\frac{q'}{p'}} \frac{\mathrm{d}t}{t} = c\|h_{k,N}^*\|_{L^{p',q'}}^{q'}.
\end{aligned}$$

(e) Conclude by showing that $h \in L^{p',q'}(\Omega)$.

**Solution:** Putting together (b), (c) and (d), and observing that $\frac{q'}{q} = q' - 1$, we have:

$$\begin{aligned}\|h_{k,N}\|^{q'}_{L^{p',q'}} = \|h^*_{k,N}\|^{q'}_{L^{p',q'}} &\le C \int_0^\infty \varphi(t) h^*_{k,N}(t)\,\mathrm{d}t \le C\|T\|\,\|\varphi\|_{L^{p,q}} \\ &\le C\|T\|\,\|h_{k,N}\|^{q'/q}_{L^{p',q'}} = C\|T\|\,\|h_{k,N}\|^{q'-1}_{L^{p',q'}}.\end{aligned}$$

Since $h_{k,N} \in L^1 \cap L^\infty(\Omega)$, $\|h_{k,N}\|_{L^{p',q'}} < \infty$ and we may divide to get

$$\|h_{k,N}\|_{L^{p',q'}(\Omega)} \le C\|T\|.$$

**Exercise 12.2. ★**
Prove that there exists a function $f \in L^1(\mathbb{R}^2)$ such that, for any $u \in \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\Delta u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}^2), \tag{$\star$}$$

$\nabla^2 u \notin L^1_{\mathrm{loc}}(\mathbb{R}^2)$.
**Hint:** consider the function

$$f(x) := \frac{1}{|x|^2 \log^2(|x|)}\, \mathbb{1}_{B_{1/2}(0)}(x);$$

then find a *radial* solution $u_0$ of ($\star$) explicitly and show that $\nabla^2 u_0 \notin L^1(B_{1/2}(0))$. Conclude for an arbitrary solution $u$ of ($\star$).

**Solution:** See Lemma 7.17 in the script.

**Exercise 12.3.**
Suppose that $T$ is the convolution operator given by a kernel $K$ satisfying the hypotheses of Theorem 7.5. Prove that, if $f \in L^1 \log L^1(\mathbb{R}^n)$, then $Tf \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for any measurable set $A \subset \mathbb{R}^n$ with $|A| < \infty$, it holds

$$\int_A |Tf(y)|\,\mathrm{d}y \le C \int_{\mathbb{R}^n} |f(y)| \log\left(e + |A| \frac{|f(y)|}{\|f\|_{L^1(\mathbb{R}^n)}}\right) \mathrm{d}y,$$

where $C > 0$ is a constant only depending on $T$.

**Hint:** the proof of this statement uses ingredients from the proof of Theorem 7.5 (the analogous theorem but for $L^p \to L^p$) and of Theorem 5.8 (the analogous theorem but for the maximal operator instead of a convolution operator). Make sure you understand both proofs and try to combine them.

**Solution:** See Theorem 7.18 in the script (beware of some typos!).

**Exercise 12.4.**

(a) Show that there is a constant $C \geq 1$ such that, for all $t > 0$,

$$\frac{1}{C} t(1 + \log^+(t)) \leq t \log(e + t) \leq Ct(1 + \log^+(t)).$$

**Solution:** Clearly if $t < 1$ we have $1 + \log^+(t) = 1$ and

$$1 = \log(e) \leq \log(e + t) \leq \log(e + 1) = C \cdot 1.$$

On the other hand, for $t \geq 1$, $1 + \log^+(t) = \log(e) + \log(t) = \log(et)$. It is clearly enough to show that $e + t \leq (et)^2$ and $et \leq (e + t)^2$, but these inequalities are straightforward:

$$(et)^2 \geq \frac{e}{2} et + \frac{e}{2} et \geq e + t$$

and

$$(e + t)^2 = e^2 + t^2 + 2et \geq et.$$

(b) Show that, for a function $f : \Omega \to \mathbb{R}$, the following four properties are equivalent:

(i) $\displaystyle\int_\Omega |f| \log(e + |f|) < \infty$.

(ii) $\displaystyle\int_\Omega |f|(1 + \log^+(|f|)) < \infty$.

(iii) $\displaystyle\int_\Omega \frac{|f|}{K} \log\left(e + \frac{|f|}{K}\right) \leq 1$ for some constant $0 < K < \infty$.

(iv) $\displaystyle\int_\Omega \frac{|f|}{K} \left(1 + \log^+\left(\frac{|f|}{K}\right)\right) \leq 1$ for some constant $0 < K < \infty$.

Moreover, show that if $|\Omega| < \infty$, then the “1+” can be removed from (ii) and (iv).

**Solution:** By part (a), (i) and (ii) are clearly equivalent. The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are done in the same way: the functions

$$\Phi_1(t) = t \log(e + t) \qquad \text{and} \qquad \Phi_2(t) = t(1 + \log^+(t))$$

are clearly increasing in $t$ and satisfy $\Phi_j(0) = 0$, so $\Phi_j(t/K) \leq \Phi_j(t)$ whenever $K \geq 1$. Therefore if we know that $\int_\Omega \Phi_j(|f|) < \infty$, by the Dominated Convergence Theorem we have that

$$\lim_{K\to\infty} \int_\Omega \Phi_j(|f|/K) = \int_\Omega \Phi_j(0) = 0$$

and therefore $\int_\Omega \Phi_j(|f|/K) \leq 1$ for all $K$ large enough.

Conversely, to prove (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii), we first show that, given $K \geq 1$, $\Phi_j(t/K) \geq c\Phi_j(t)$ for all $t > 0$, where $c$ is a constant depending only on $K$. This is equivalent to showing that

$$\frac{\Phi_j(t/K)}{t/K} \geq cK\frac{\Phi_j(t)}{t},$$

or equivalently, that

$$\frac{\Phi_j(Ks)}{Ks} \leq C\frac{\Phi_j(s)}{s}.$$

By part (a), it suffices to show this for $\Phi_1$, and this is clear:

$$\log(e+Ks) \leq \log(Ke+Ks) \leq \log(K) + \log(e+s) \leq (1+\log(K))\log(e+s).$$

Finally, to prove the last assertion, first notice the following inequality for every $M, K \geq 1$:

$$\begin{aligned}
\int_\Omega \frac{|f|}{MK}\left(1+\log^+\left(\frac{|f|}{MK}\right)\right) &= \int_\Omega \frac{|f|}{MK}\left(1+\log^+\left(\frac{|f|}{MK}\right)\right)\mathbb{1}_{|f|<Ke} + \frac{|f|}{MK}\left(1+\log^+\left(\frac{|f|}{MK}\right)\right)\mathbb{1}_{|f|\geq Ke} \\
&\leq \frac{e}{M}\left(1+\log^+\left(\frac{e}{M}\right)\right)|\Omega| + \int_\Omega \frac{|f|}{MK}\left(\log^+\left(\frac{|f|}{K}\right)+\log^+\left(\frac{|f|}{MK}\right)\right) \\
&\leq 2\frac{e}{M}|\Omega| + \frac{2}{M}\int_\Omega \frac{|f|}{K}\left(\log^+\left(\frac{|f|}{K}\right)\right).
\end{aligned}$$

Setting $K = M = 1$ shows that

$$\int_\Omega |f|\log^+|f| < +\infty \qquad \Longrightarrow \qquad \int_\Omega |f|\left(1+\log^+|f|\right) < +\infty;$$

setting $K$ such that (iv) without the "1+" holds and $M := 2e|\Omega| + 2$ we also obtain that

$$\int_\Omega \frac{|f|}{MK}\left(1+\log^+\left(\frac{|f|}{MK}\right)\right) \leq \frac{2e|\Omega|+2}{M} = 1.$$

(c) Let $\Phi : [0,\infty) \to [0,\infty)$ be a nonnegative, strictly increasing, continuous and convex function satisfying $\Phi(0) = 0$, and define

$$\|f\|_\Phi := \inf\left\{0 < K < \infty : \int_\Omega \Phi(|f|/K)\,\mathrm{d}x \leq 1\right\},$$

where the infimum is understood to be $+\infty$ if no such $K$ exists. Show that the set of (equivalence classes of almost everywhere equal) functions

$$L_\Phi(\Omega) := \{f : \Omega \to \mathbb{R} \text{ measurable s.t. } \|f\|_\Phi < \infty\}$$

is a vector space and that $\|\cdot\|_\Phi$ defines a norm on $L_\Phi(\Omega)$. Show moreover that $L_\Phi(\Omega)$ is complete with respect to this norm.

**Solution:** It is clear that $f \in L_\Phi \Rightarrow \lambda f \in L_\Phi$ with $\|\lambda f\|_\Phi = |\lambda|\,\|f\|_\Phi$ for each $\lambda \in \mathbb{R}^*$, and that $\|0\|_\Phi = 0$. Moreover, $\|f\|_\Phi = 0$ implies that $\Phi(|f|) = 0$ a.e., so $|f| = 0$ a.e. too. It only remains to show the triangle inequality:

$$\begin{aligned}\int_\Omega \Phi\left(\frac{f+g}{\|f\|_\Phi+\|g\|_\Phi}\right) &= \int_\Omega \Phi\left(\frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}\frac{f}{\|f\|_\Phi}+\frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}\frac{g}{\|g\|_\Phi}\right)\\ &\le \frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}\int_\Omega \Phi\left(\frac{f}{\|f\|_\Phi}\right)+\frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}\int_\Omega \Phi\left(\frac{g}{\|g\|_\Phi}\right)\\ &\le \frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}+\frac{\|f\|_\Phi}{\|f\|_\Phi+\|g\|_\Phi}=1.\end{aligned}$$

Here we have used Jensen's inequality and the fact that the infimum defining $\|\cdot\|_\Phi$ is a minimum, which is a consequence of Monotone Convergence.

For the completeness: as in the usual case, given a Cauchy sequence $(f_j)$ for $\|\cdot\|_\Phi$, we choose a subsequence $(f_{k_j})$ such that

$$A := \sum_{j=1}^\infty \|f_{k_{j+1}}-f_{k_j}\|_\Phi < \infty$$

and show that it converges. Let

$$g_N := \sum_{j=1}^N |f_{k_{j+1}}-f_{k_j}| \quad \xrightarrow{N\to\infty} \quad g := \sum_{j=1}^\infty |f_{k_{j+1}}-f_{k_j}|$$

and observe that $\|g_N\|_\Phi \le A$ for all $N$ by the triangle inequality. Since $\Phi$ is increasing, by the monotone convergence theorem,

$$\int_\Omega \Phi(g/A)\,\mathrm{d}x = \lim_{N\to\infty}\int_\Omega \Phi(g_N/A)\,\mathrm{d}x \le 1,$$

where we are extending $\Phi(\infty)=\infty$. Notice that $\Phi(t)\to\infty$ as $t\to\infty$—otherwise, if $\Phi\le C$, by convexity $\Phi(t)=\Phi\left(\frac{t}{M}M+\left(1-\frac{t}{M}\right)0\right)\le \frac{t}{M}\Phi(M)+\left(1-\frac{t}{M}\right)\Phi(0)\le tC/M$ and sending $M\to\infty$ we get that $\Phi(t)=0$.

Therefore $\Phi(g/A)<\infty$ almost everywhere, so $g<\infty$ almost everywhere too, and hence the sum $f_{j_1}+\sum_{j=1}^\infty (f_{k_{j+1}}-f_{k_j})$ converges almost everywhere to a function $f:\Omega\to\mathbb{R}$. Finally, let

$$A_N := \sum_{j=N}^\infty \|f_{k_{j+1}}-f_{k_j}\|_\Phi \xrightarrow{N\to\infty} 0,$$

so that

$$\int_\Omega \Phi\left(\sum_{j=N}^\infty |f_{k_{j+1}}-f_{k_j}|/A_N\right) = \lim_{M\to\infty}\int_\Omega \Phi\left(\sum_{j=N}^M |f_{k_{j+1}}-f_{k_j}|/A_N\right)\le 1$$

arguing as above. Therefore, since

$$|f-f_{k_N}| = \left|\sum_{j=N}^\infty (f_{k_{j+1}}-f_{k_j})\right| \le \sum_{j=N}^\infty |f_{k_{j+1}}-f_{k_j}|,$$

also $\|f-f_{k_N}\|_\Phi \le A_N$ and hence $f_{k_j}\to f$ in $L_\Phi(\Omega)$.

(d) Deduce that the space

$$L\log L(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \int_\Omega |f| \log(e + |f|) < \infty \right\} = L^1(\Omega) \cap \left\{ f : \int_\Omega |f| \log^+ |f| < \infty \right\}$$

can be given a natural structure of Banach space.

**Solution:** By part (b), the set of functions $f \in L\log L(\Omega)$ coincides with the set of functions belonging to the linear space $L_\Phi(\Omega)$ with $\Phi = \Phi_1$ or $\Phi_2$. An easy computation shows that both $\Phi_1$ and $\Phi_2$ are increasing and convex, so we may apply part (c). This gives the so-called Luxembourg norm for the $L\log L$ Orlicz space.