Exercise 13.1. ★

(a) Show that for every $p \in [1,\infty]$ and $q \in [1,\infty]$, the space $L^p(\mathbb{R}^n, \ell^q)$ defines a Banach space.

(b) Show that for every $p \in (1,\infty)$ and $q \in (1,\infty)$, the dual of $L^p(\mathbb{R}^n, \ell^q)$ is $L^{p'}(\mathbb{R}^n, \ell^{q'})$.

Solution: See Proposition 8.4 in the script.

Exercise 13.2. ★

The goal of this exercise is to prove Khinchine's inequality and see an application to Fourier analysis. This states the following: let $1 \le p < \infty$; then there exists a constant $C = C(p) > 0$ such that, for any $N \in \mathbb{N}$ and any $a_1, \dots, a_N \in \mathbb{C}$, it holds that

$$C^{-1}\left(\sum_{j=1}^N |a_j|^2\right)^{p/2} \le \mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^p\right] \le C\left(\sum_{j=1}^N |a_j|^2\right)^{p/2},$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the uniformly distributed random variable $(\epsilon_j)_{j=1,\dots,N} \in \{-1,+1\}^N$. In other words, it is the average of the expression inside the $[\cdot]$ over the 2^N possible choices of signs.

(a) Prove the upper bound of Khinchine's inequality.

Hint: you may use the following fact from probability: if N, (a_j) and (ϵ_j) are as above, then for every $\lambda > 0$,

$$\mathbb{P}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right| > \lambda\left(\sum_{j=1}^N |a_j|^2\right)^{1/2}\right] \le 4e^{-\lambda^2/2}.$$

Combine this with the usual formula for the integral using upper level sets, but interchanging measures/integrals by probabilities/expectations.

Solution: We have that

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^p\right] &= \int_0^\infty \mathbb{P}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^p > \mu\right] \mathrm{d}\mu \\
&= \left(\sum_{j=1}^N |a_j|^2\right)^{p/2} \int_0^\infty \mathbb{P}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^p > \lambda^p\left(\sum_{j=1}^N |a_j|^2\right)^{p/2}\right] p\lambda^{p-1}\mathrm{d}\lambda \\
&\le p\left(\sum_{j=1}^N |a_j|^2\right)^{p/2} \int_0^\infty 4e^{-\lambda^2/2}\,\lambda^{p-1}\mathrm{d}\lambda \\
&= C(p)\left(\sum_{j=1}^N |a_j|^2\right)^{p/2},
\end{aligned}$$

where we have used the change of variable $\mu = \left(\sum_{j=1}^N |a_j|^2\right)^{p/2} \lambda^p$.

(b) Prove the lower bound of Khinchine's inequality.

Hint: bound the expectation with $p = 2$ by the expectation with a higher p and a lower p by using Hölder's inequality, and then estimate the term with the higher p by using part (a).

Solution: Observe that, when $p = 2$, by the independence of the ϵ_j,

$$\mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^2\right] = \mathbb{E}\left[\sum_{j,k=1}^N \epsilon_j \epsilon_k a_j \overline{a_k}\right] = \sum_{j=1}^N a_j \overline{a_j} = \sum_{j=1}^N |a_j|^2.$$

Therefore, by Hölder's inequality, the first inequality is trivial if $p \geq 2$. When $1 \leq p < 2$ we compute, using Hölder's inequality for the expectation,

$$\sum_{j=1}^N |a_j|^2 = \mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^2\right] \leq \mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^p\right]^{1/p} \mathbb{E}\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^{p'}\right]^{1/p'},$$

and apply part (a) for $p' > 2$:

$$\left[\left|\sum_{j=1}^N \epsilon_j a_j\right|^{p'}\right]^{1/p'} \leq C\left[\left(\sum_{j=1}^N |a_j|^2\right)^{p'/2}\right]^{1/p'} = C\left(\sum_{j=1}^N |a_j|^2\right)^{1/2}.$$

Simplifying gives the lower bound of the Khinchine inequality.

(c) Show that, given $p > 2$, there exist two functions $f, g : \mathbb{S}^1 \to \mathbb{C}$ such that the Fourier coefficients of f and g have the same absolute values, but such that $f \notin L^p(\mathbb{S}^1)$ and $g \in L^p(\mathbb{S}^1)$. Deduce that there is no characterization of belonging to L^p based on summability properties of the Fourier coefficients.

Solution: For a given $p > 2$, choose some $f \in L^2(\mathbb{S}^1) \setminus L^p(\mathbb{S}^1)$ and let $(\hat{f}_j)_{j\in\mathbb{Z}}$ be its Fourier coefficients. For any $N \in \mathbb{N}$ and $0 \leq \theta < 2\pi$, apply Khinchine's inequality to the numbers $\hat{f}_j e^{ij\theta}$, $|j| \leq N$:

$$\mathbb{E}\left[\left|\sum_{|j|\leq N} \epsilon_j \hat{f}_j e^{ij\theta}\right|^p\right] \leq \left(\sum_{|j|\leq N} |\hat{f}_j|^2\right)^{p/2}.$$

Integrating over θ and using the linearity of expectation, we get

$$\mathbb{E}\left[\int_0^{2\pi} \left|\sum_{|j|\leq N} \epsilon_j \hat{f}_j e^{ij\theta}\right|^p \mathrm{d}\theta\right] = \int_0^{2\pi} \mathbb{E}\left[\left|\sum_{|j|\leq N} \epsilon_j \hat{f}_j e^{ij\theta}\right|^p\right] \mathrm{d}\theta \leq 2\pi \left(\sum_{|j|\leq N} |\hat{f}_j|^2\right)^{p/2}.$$

Therefore, for some choice of signs $\epsilon_{-N}^{(N)}, \dots, \epsilon_N^{(N)}$ depending on N, we have that the function

$g_N(\theta) := \sum_{|j|\le N} \epsilon_j^{(N)} \hat{f}_j e^{ij\theta}$ satisfies

$$\|g_N\|_{L^p(\mathbb{S}^1)} = \left(\int_0^{2\pi} |g_N(\theta)|^p \,\mathrm{d}\theta\right)^{1/p} \le (2\pi)^{1/p} \left(\sum_{|j|\le N} |\hat{f}_j|^2\right)^{1/2} \le (2\pi)^{1/p-1/2} \|f\|_{L^2(\mathbb{S}^1)}.$$

Now the sequence (g_N) is bounded in $L^p(\mathbb{S}^1)$ and therefore, after taking a subsequence, it converges weakly in L^p to some function $g \in L^p(\mathbb{S}^1)$—using that for $2 < p < \infty$ the L^p spaces are reflexive. On the other hand, by weak convergence, the Fourier coefficients of g satisfy

$$\hat{g}_j = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-ij\theta}\,\mathrm{d}\theta = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} g_N(\theta)e^{-ij\theta}\,\mathrm{d}\theta = \lim_{N\to\infty} \epsilon_j^{(N)}\hat{f}_j,$$

and therefore $|\hat{g}_j| = |\hat{f}_j|$ for each j (notice that this implies that the signs $\epsilon_j^{(N)}$ eventually stabilize for each j such that $\hat{f}_j \neq 0$).

Exercise 13.3.
Let $1 < p < \infty$ and suppose that $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear transformation which commutes with translations. Show that there exists a function $m \in L^\infty(\mathbb{R}^n)$ such that

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n$$

whenever $f \in L^2 \cap L^p$.

Hint: show this first for $p = 2$. For general p, argue by duality to conclude that T is also of type (p', p') and then apply the case $p = 2$.

Solution: For $v \in \mathbb{R}^n$, denote $\tau_v f(x) := f(x - v)$. We first claim that for any $f \in L^p(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$, $T(f \star g) = (Tf) \star g$.

Recall that for $f \in \mathcal{S}'$ and $u \in \mathcal{E}'$, we characterized $f \star u \in \mathcal{S}'$ as

$$\langle f \star u, \varphi\rangle = \langle u, \check{f} \star \varphi\rangle \qquad \forall \varphi \in \mathcal{S}.$$

It is clear that $\check{f} \star \varphi \in C^\infty(\mathbb{R}^n)$, and if $u_j \rightharpoonup u$ in $\mathcal{E}'$, then by the above expression $f \star u_j \rightharpoonup f \star u$ in $\mathcal{S}'$. Given any function $g \in C_c^\infty(\mathbb{R}^n)$, the sequence

$$u_N := \frac{1}{N^n}\sum_{k\in\mathbb{Z}^n} g(k/N)\delta_{k/N} \in \mathcal{E}'(\mathbb{R}^n)$$

clearly converges in $\mathcal{E}'$ to g. On the other hand, if $f \in L^p(\mathbb{R}^n)$, then $f \star u_N \in L^p(\mathbb{R}^n)$ with equibounded norm, because

$$\langle f \star u_N, \varphi\rangle = \langle u_N, \check{f} \star \varphi\rangle \le \|u_N\|_{\mathcal{M}} \|\check{f} \star \varphi\|_{C^0} \le (2K)^n \|g\|_{C^0}\|f\|_{L^p}\|\varphi\|_{L^{p'}} \qquad \forall \varphi \in \mathcal{S}$$

if g is supported in $(-K, K)^n$ for $K \in \mathbb{N}$. This together with Exercise 2.6(a) gives that also $f \star u_N \rightharpoonup f \star g$ in L^p.

Now writing $u_N = \sum_{j\in\mathbb{Z}^n} a_j\delta_{j/N}$, we have for each N (note that all the sums are finite because g has compact support):

$$T\left(f\star\sum_{j\in\mathbb{Z}^n} a_j\delta_{j/N}\right) = \sum_{j\in\mathbb{Z}^n} a_jT\left(\tau_{j/N}f\right) = \sum_{j\in\mathbb{Z}^n} a_j\tau_{j/N}(Tf) = (Tf)\star\left(\sum_{j\in\mathbb{Z}^n} a_j\delta_{j/N}\right),$$

which means that $T(f\star u_N) = (Tf)\star u_N$. When we send $N\to\infty$, the right hand side converges to $(Tf)\star g$ in $\mathcal{S}'$ by the above discussion, whereas the left hand side converges to $T(f\star g)$ weakly in L^p because bounded linear operators send weakly converging sequences to weakly converging sequences. The claim now follows for $g\in C_c^\infty(\mathbb{R}^n)$, and for a general Schwartz function (even for an L^1 function) we obtain it by approximation and the continuity of the convolution $L^p\times L^1\to L^p$.

Now we show that T is also bounded in the dual space: if $f,g\in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}\langle g, Tf\rangle &= \int_{\mathbb{R}^n} g(x)Tf(x)\,\mathrm{d}x = (\check{g}\star Tf)(0) = T(\check{g}\star f)(0) = (T\check{g}\star f)(0) = \langle T\check{g},\check{f}\rangle\\ &\le \|T\check{g}\|_{L^p}\,\|\check{f}\|_{L^{p'}} \le \|T\|\,\|\check{g}\|_{L^p}\,\|\check{f}\|_{L^{p'}} = \|T\|\,\|g\|_{L^p}\,\|f\|_{L^{p'}}.\end{aligned}$$

Taking the supremum over g gives that $\|Tf\|_{L^{p'}}\le\|T\|\,\|f\|_{L^{p'}}$, so by density T extends to a linear bounded operator $L^{p'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)$. Now the Riesz–Thorin interpolation theorem gives that T is also bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ (with the same norm), so for the rest of the proof it will be enough to characterize T acting on this space.

Denote by $M : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ the operator $M\hat{f} = \widehat{Tf}$, which is well-defined thanks to Plancherel and $\|M\| = \|T\|_{L^2\to L^2}$. We claim that $fMg = M(fg)$ for all $f,g\in\mathcal{S}(\mathbb{R}^n)$: since the Fourier transform is an isomorphism in $\mathcal{S}$, it is enough to show

$$M(\hat{f}\hat{g}) = (2\pi)^{-n/2}M\left(\widehat{f\star g}\right) = (2\pi)^{-n/2}\widehat{T(f\star g)} = (2\pi)^{-n/2}\widehat{f\star Tg} = \hat{f}\widehat{Tg} = \hat{f}M\hat{g}.$$

By density we also get $fMg = M(fg)\in L^2$ if $f\in L^2$ and $g\in\mathcal{S}$.

Now fix a function $\chi\in C_c^\infty(B_2)$ with $\chi\equiv 1$ on B_1 and $\chi\ge 0$ everywhere. For $R>0$, let $\chi_R(x) := \chi(x/R)$ and $m_R := M\chi_R\in L^2(\mathbb{R}^n)$. If $f\in L^1\cap L^2$, we can write $f = \mathrm{sgn}(f)\sqrt{|f|}\cdot\sqrt{|f|}$, with $\sqrt{|f|}\in L^2$, and therefore

$$\begin{aligned}\int fm_R &= \int \mathrm{sgn}(f)\sqrt{|f|}\sqrt{|f|}\cdot M\chi_R = \int\mathrm{sgn}(f)\sqrt{|f|}\cdot M(\sqrt{|f|}\chi_R)\\ &\le \left\|\mathrm{sgn}(f)\sqrt{|f|}\right\|_{L^2}\left\|M(\sqrt{|f|}\chi_R)\right\|_{L^2}\\ &\le \|T\|\left\|\sqrt{|f|}\right\|_{L^2}^2 = \|T\|\,\|f\|_{L^1}.\end{aligned}$$

Taking the supremum over such f, and using the duality of L^1 and L^∞, we obtain that $m_R\in L^\infty$ with $\|m_R\|_{L^\infty}\le\|T\|$. Since $L^1(\mathbb{R}^n)$ is separable, by Banach–Alaoglu, for a sequence $R_j\to\infty$ we have that $m_{R_j}\overset{*}{\rightharpoonup} m$ in L^∞, for some $m\in L^\infty(\mathbb{R}^n)$.

We also have that for any $f\in L^2$, $\chi_Rf\to f$ in L^2 as $R\to\infty$, so

$$M(f) = \lim_{R\to\infty} M(\chi_Rf) = \lim_{R\to\infty} fM(\chi_R) = \lim_{R\to\infty} fm_R$$

in L^2. Therefore, for any $g \in L^2(\mathbb{R}^n)$ it holds that

$$\int gM(f) = \lim_{j\to\infty} \int gfm_{R_j} = \int gfm,$$

and therefore $Mf = mf$, which is what we wanted to show. (Notice that actually the whole sequence $m_R \overset{*}{\rightharpoonup} m$: for any $f \in L^1 \cap L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$,

$$\int gfm_R \xrightarrow{R\to\infty} \int gM(f) = \int gfm,$$

and we can write any L^1 function in the form gf as above.)