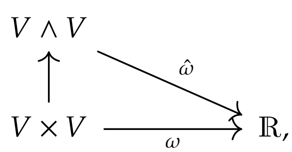
Exercises 19.02.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (1.1.2)**.** *Let V be a real vector space. Show that the datum of a antisymmetric bilinear map $\omega : V \times V \to \mathbb{R}$ is equivalent to that of a linear map $\hat{\omega} : V \wedge V \to \mathbb{R}$ making the following diagram commute*



where the vertical map sends (v, w) to $v \wedge w$.

Exercise 2 (1.1.4)**.** *Show that every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$ for some n.*

Exercise 3. *Let A be a $(2n \times 2n)$-matrix, and write it as*

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where X, Y, Z, and W are all $(n \times n)$-matrices. Write down the equations on these $(n \times n)$-matrices that are equivalent to $A \in \mathrm{Symp}(n)$.

Exercise 4 (1.1.6)**.** *Show that* $\mathrm{Symp}(1)$ *is isomorphic as a (Lie) group to* $\mathrm{SL}(2; \mathbb{R})$.

Exercise 5 (1.2.1)**.** *Let M be a $2n$-dimensional manifold. Show that a 2-form $\omega \in \Omega^2(M)$ is nondegenerate if and only if $\omega^n \in \Omega^{2n}(M)$ is nowhere vanishing.*

Exercise 6 (1.2.4)**.** *Show that a closed manifold never admits an exact nondegenerate 2-form.*

Exercise 7 (1.2.7)**.** *Let L be a n-dimensional manifold, and let $U \subseteq L$ an open such that $\pi : T^*L \to L$ is trivial over U, i.e. $\pi^{-1}(U)$ is isomorphic as a vector bundle to $U \times \mathbb{R}^n$. We define the* ***tautological 1-form*** *on $\pi^{-1}(U)$ as*

$$\lambda_0|_{\pi^{-1}(U)} = \sum_{i=1}^{n} p_i dq_i, \tag{1}$$

where the q_i's are coordinates in U and the p_i's are the canonical coordinates of $\mathbb{R}^n$.

Show that (1) defines a global 1-form λ_0 on T^*L. Furthermore, show that λ_0 is the unique 1-form on T^*L such that

$$\sigma^*\lambda_0 = \sigma \qquad \forall \sigma \in \Omega^1(L),$$

where, on the LHS, we see σ as a map $L \longrightarrow T^*L$.

Exercise 8 (1.2.13)**.** *Read the proof of Darboux's theorem in either McDuff and Salamon's Introduction to Symplectic Topology (Theorem 3.2.2 in the third edition) or da Silva's Lectures on Symplectic Geometry (Theorem 8.1).*

Exercise 9 (1.3.1)**.** *Let H be a (time-dependent) function on a symplectic manifold (M, ω). Show that there is a unique (time-dependent) vector field X_H such that*

$$\iota_{X_H}\omega = -dH.$$

Exercise 10 (1.3.5)**.** *Let H be an autonomous Hamiltonian on a symplectic manifold (M, ω), i.e. a function $H : M \longrightarrow \mathbb{R}$. Show that its Hamiltonian flow $\{\varphi_H^t\}$ preserves its level sets, i.e. $H \circ \varphi_H^t = H\ \forall t$.*

Exercise 11. *Let (M, ω) is a closed symplectic manifold, and let H and G be Hamiltonians on M. Let also ψ be a* ***symplectomorphism*** *of M, i.e. a diffeomorphism such that $\psi^*\omega = \omega$.*

(1) *Show that the Hamiltonian $H\#G$ defined as*

$$(H\#G)_t := H_t + G_t \circ (\varphi_H^t)^{-1}$$

generates the isotopy $\{\varphi_t^H \circ \varphi_t^G\}$.

(2) *Show that the Hamiltonian $\overline{H}$ defined as*

$$\overline{H}_t := -H_t \circ \varphi_t^H$$

generates the isotopy $\{(\varphi_t^H)^{-1}\}$.

(3) *Show that the Hamiltonian ψ^*H defined as*

$$(\psi^*H)_t := H_t \circ \psi$$

generates the isotopy $\{\psi^{-1} \circ \varphi_t^H \circ \psi\}$.

(4) *Conclude that* $\mathrm{Ham}(M)$*, the set of Hamiltonian diffeomorphisms of M, is a normal subgroup of* $\mathrm{Symp}(M)$*, the group of symplectomorphisms of M.*