EXERCISES 30.04.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (5.5.4, Lemma 5.5.3). *Let X and Y be Banach spaces. Let $F : X \to Y$ be a continuous map that is differentiable at 0. Write*

$$F(x) = F(0) + L(x) + N(x)$$

for $L := (dF)_0$. Suppose that there exists $G : Y \to X$ continuous such that

- *(i)* $L \circ G = \mathbb{1}_Y$;
- *(ii)* $||GN(x) - GN(y)|| \leq C\,(||x|| + ||y||)\,||x - y||$ *for all* $x, y \in B(0, r)$;
- *(iii)* $||GF(0)|| \leq \frac{\varepsilon}{2}$,

where $C, r > 0$ *and* $\varepsilon := \min\{r, \frac{1}{5C}\}$.

- *(1) Show that the map* $\varphi(x) = G(L(x) - F(x))$ *is a contraction on the ball* $B(0, \varepsilon)$. *Conclude that there is a unique* $\alpha \in B(0, \varepsilon)$ *such that* $F(\alpha) = 0$.
- *(2) Show that* $||\alpha|| \leq 2||GF(0)||$.
- *(3) Consider the case where* $X = Y = \mathbb{R}$ *and* $F = f : \mathbb{R} \to \mathbb{R}$ *is twice differentiable at 0 with* $f'(0) \neq 0$. *Show that* $G(x) = \frac{x}{f'(0)}$ *satisfies the above hypotheses — for some* $C, r > 0$ *— if* $|f(0)|$ *is small enough (when compared to* $|f'(0)|$*), or if* $\lim_{x \to \pm\infty} \frac{f(x)}{x^2} = 0$. *Conclude that the classical Newton method converges in those cases.*

Exercise 2 (5.6.1). *Show that a proper, injective immersion is an embedding. Show that any immersion into a 1-dimensional manifold is injective.*