Exercises 07.05.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (6.4.2)**.** *Let $f : [0,1] \to \mathbb{R}$ be differentiable with $\int_0^1 f(t)dt = 0$. Show that, for all $p \geq 1$, we have that*

$$\int_0^t |f|^p dt \leq \int_0^1 |f'|^p dt$$

Hint: *Use the fundamental theorem of calculus and the fact that $\int_0^1 f dt = 0$ to write f as a double integral of f'. Conclude by using the triangle inequality.*

Exercise 2. *On the n-sphere $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^{n+1}$, consider the height function $f(x) = x_0$. Let g be the restriction of the Euclidean metric on S^n. Show that (f, g) is Morse–Smale, and compute $HM_\bullet(f, g)$.*

Exercise 3. *On $S^2 = \{x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$, consider instead the function $f(x, y, z) = y^2 + 2z^2$. Denote by F the function $\mathbb{R}P^2 \to \mathbb{R}$ it induces under the identification $\mathbb{R}P^2 = S^2/\{\pm 1\}$. Let g be the restriction of the Euclidean metric to S^2, and let G be the metric it induces on $\mathbb{R}P^2$.*

(1) Show that f (and thus F) is a self-indexing Morse function, i.e. the value of f at a critical point is the index of the point.

(2) Show that (f, g) and (F, G) are Morse–Smale, and compute their Morse homologies.

Exercise 4. *Let $f : \mathbb{T}^2 \to \mathbb{R}$ be the height function of the torus under its usual embedding in $\mathbb{R}^3$, and let g be the restriction of the Euclidean metric to $\mathbb{T}^2$ under that embedding.*

*(1) Show that f is Morse but that (f, g) is **not** Morse–Smale.*

(2) Perturb f so that (f, g) is indeed Morse–Smale, and compute the resulting Morse homology.

Hint: *To perturb f, simply tilt the torus in $\mathbb{R}^3$ appropriately.*