EXERCISES 12.03.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (2.7.4). *Using Proposition 2.3.3 and Exercise 2 of 26.02.2025, show that $\mathscr{A}_H$ has finitely many critical points whenever H is nondegenerate.*

Exercise 2 (Hard). *Let $U \subseteq \mathbb{T}^n$ be open, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the flat n-torus. We define a topology on $C^k(U, \mathbb{R}^m)$ as follows. For any compact $K \subseteq U$ and any $r \leq k$, take the pseudo-norm on $C^k(U, \mathbb{R}^m)$ defined by*

$$||f||_K^r := \sup_{x \in K} |(\nabla^r f)(x)|,$$

*where $\nabla^r f : U \to \mathbb{R}^{n^r m}$ is the r-th iterate of the gradient of f, i.e. its coordinates are all the possible derivatives of f of order r. Then, a subset $\mathcal{W}$ of $C^k(U, \mathbb{R}^m)$ is **open** if and only if, for every $f \in \mathcal{W}$, there is a $\varepsilon > 0$, a compact $K \subseteq U$, and some $k_0 < \min\{k+1, +\infty\}$ such that*

$$\left\{ g \in C^k(U, \mathbb{R}^m) \,\middle|\, ||f - g||_K^r < \varepsilon \ \forall r \leq k_0 \right\} \subseteq \mathcal{W}.$$

Show that this makes $C^k(U, \mathbb{R}^m)$ into a Fréchet vector space and that the topology correspond to the C^k_{loc} one defined in class.

When $U = \mathbb{T}^n$ and $k < +\infty$, use the pseudo-norms $|| \cdot ||_K^r$ to construct a norm $|| \cdot ||_{C^k}$ on $C^k(\mathbb{T}^n, \mathbb{R}^m)$ making it into a Banach space. For general U — but still $k < +\infty$ — construct a similar norm on the subspace $C^k_0(U, \mathbb{R}^m)$ of compactly supported functions. Show that the topology induced by these norms are the same as the one defined above.

Exercise 3 (2.7.8). *Construct a metric d_∞ on $\mathscr{L}M$ inducing the C^∞ topology.*