Exercises 19.03.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (2.7.12; Lemma 2.7.11). *Let $g : X \to \mathbb{R}$ be continuous, where X is a complete metric space with metric d. Take $x_0 \in X$ and $\varepsilon_0 > 0$. Prove that there exist $x \in X$ and $\varepsilon \in (0, \varepsilon_0]$ such that*

$$\begin{cases} d(x, x_0) \leq 2\varepsilon; \\ \varepsilon g(x) \geq \varepsilon_0 g(x_0); \\ g(y) \leq 2g(x) \qquad \forall y \in B(x, \varepsilon). \end{cases}$$

Exercise 2 (Some complex geometry). *Consider the complex curve*

$$\Sigma' := \{y^2 = 4x^3 - x - 1\} \subseteq \mathbb{C}^2$$

and the holomorphic map

$$\begin{aligned} u'_\alpha : \Sigma' &\longrightarrow \mathbb{C}^2 \\ (x, y) &\longmapsto (\alpha^2 x, \alpha^3 y) \end{aligned}$$

for $\alpha \in (0, 1]$.

(1) Show that Σ' extends to a complex curve Σ in $\mathbb{C}P^2$ and that u'_α extend to a holomorphic map $u_\alpha : \Sigma \to \mathbb{C}P^2$.

(2) Show that Σ is topologically a torus.
Hint: *Use the equation defining Σ' to construct a 2:1 branched cover to $\mathbb{C}$ with 3 branch points, and extend it to $\Sigma \to \mathbb{C}P^1 = S^2$ (now with 4 branch points).*

(3) Illustrate why the images of u_α converge to the wedge product of a torus with a sphere as $\alpha \to 0$.

Exercise 3 (3.1.1). *Prove that* $\mathrm{Symp}(n)$ *is a Lie group of dimension $n(2n + 1)$.*
Hint: *Consider the map $A \mapsto A^T J_0 A$.*

Exercise 4 (3.1.9)**.** *For $t \geq 0$ and A, a positive definite symmetric $n \times n$ matrix, define A^t to be the unique $n \times n$ matrix such that*

$$Av = \lambda v \qquad \Longleftrightarrow \qquad A^t v = \lambda^t v.$$

Show that A^t is positive definite and symmetric for all t. If A is furthermore symplectic, show that A^t also is.

Exercise 5. *Using the description of the generalized eigenspaces of a symplectic matrix of Corollary 3.1.7, show directly — that is, without using path connectedness — that every such matrix has determinant 1.*