EXERCISES 26.03.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (3.2.4)**.** *Check that if H is autonomous and $x \in \mathrm{Crit}(H)$, then the path of symplectic matrices associated to x and a trivialization $\Psi : T_x M \to \mathbb{R}^{2n}$ such that $\Psi^* J_0 = J_x$ and $\Psi^* \omega_0 = \omega_x$ is given by*

$$A(t) = e^{tJ_0 \mathrm{Hess}_x},$$

where Hess_x *is the symmetric matrix defined via* $(\mathrm{Hess}\, H)_x(v, w) = (\mathrm{Hess}_x \Psi(v)) \cdot \Psi(w)$.

Conclude that if x is nondegenerate and and $|\mathrm{Hess}_x|^{op} < 2\pi$, then 1 is not an eigenvalue of $A(1)$.

Hint for the first part: *Write A using Darboux coordinates, and show that it solves the appropriate differential equation.*

Exercise 2 (3.2.6)**.** *Let*

$$U = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Check that $A = US$ is symplectic and that $\rho(A) \neq \det_{\mathbb{C}} U$.

Exercise 3. *Verify that the path defined by*

$$A(t) = \begin{pmatrix} 1 + 4\pi^2 t^2 & 2\pi t \\ 2\pi t & 1 \end{pmatrix}$$

is in $\mathcal{S}$, and compute its Maslov index.

Exercise 4 (3.3.3)**.** *Show that if S is a 2×2 invertible symmetric matrix with $|S|^{op} < 2\pi$, then*

$$\mu\left(t \mapsto e^{tJ_0 S}\right) = \mathrm{Ind}(S) - 1,$$

where $\mathrm{Ind}(S)$ *is the number of negative eigenvalues of S.*

Hint: $\mathrm{Ind}(S)$ *only depends on the connected component that S belongs to in the space of invertible symmetric matrices and μ is invariant under homotopies through paths starting at $\mathbb{1}$ and ending in* $\mathrm{Symp}(n)^*$. *Use this to reduce to a choice of S that is easy to compute.*