EXERCISES 02.04.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

Exercise 1 (4.5.1)**.** *Let $J : M \to \mathrm{End}(TM)$ and $X : M \to TM$ be sections of their respective bundles, and let Y be some tangent vector to M. Show that*

$$dJ(Y) \cdot X = d(JX)(Y) - J \cdot dX(Y).$$

Exercise 2 (4.5.4)**.** *Show that the equation*

$$\frac{\partial Y}{\partial t} = J_0 S^{\pm} Y$$

is the linearization of Hamilton's equation $\dot{z}_{\pm} = X_H(z)$ at $z_{\pm} = \lim_{s\to\pm\infty} u(s)$. Here, $S^{\pm}$ is the limit as $s \to \pm\infty$ of the operator S defined by the linearized Floer operator $d\mathscr{F}_u = \bar{\partial} + S$ at u, where $\bar{\partial}$ is the Cauchy-Riemann operator.

Exercise 3 (Some basic Fredholm theory)**.** *A continuous linear map $L : E \to F$ between Banach spaces is called* Fredholm *if* Ker L *and* Coker L *are finite dimensional. In that case, we define its* index *as*

$$\mathrm{Ind}\, L := \dim \mathrm{Ker}\, L - \dim \mathrm{Coker}\, L.$$

Show that the following holds.

(1) *If E and F have finite dimension, then* $\mathrm{Ind}\, L = \dim E - \dim F$.

(2) *If $L' : E' \to F'$ is another Fredholm operator, then $L \oplus L' : E \oplus E' \to F \oplus F'$ is Fredholm and* $\mathrm{Ind}(L \oplus L') = \mathrm{Ind}\, L + \mathrm{Ind}\, L'$.

(3) *If H has finite dimension, then $L \otimes \mathbb{1}_H$ is Fredholm and* $\mathrm{Ind}(L \otimes \mathbb{1}_H) = (\dim H)\, \mathrm{Ind}\, L$.

(4) *If $L' : F \to G$ is another Fredholm operator, then $L' \circ L : E \to G$ is Fredholm and* $\mathrm{Ind}(L' \circ L) = \mathrm{Ind}\, L + \mathrm{Ind}\, L'$.