EXERCISES 16.04.2025

Optional exercises for the lecture *Introduction to Floer homology* (401-3584-25L)
Semester: Spring 2025
Lecturer: Dr. Jean-Philippe Chassé

This exercise series is less directly related to what has been done in class, but it serves to give a better understanding of what a Fredholm operator is.

Exercise 1. *Let $L : E \longrightarrow F$ be linear and continuous between Banach spaces. Show that L is Fredholm if and only if there exists $L' : F \longrightarrow E$ linear and continuous such that $L \circ L' - \mathbb{1}_F$ and $L' \circ L - \mathbb{1}_E$ have finite rank, i.e. their image is finite dimensional.*

Check that L' may be chosen so that $L \circ L' \circ L = L$. Use Exercise 3 of 02.04.25 to conclude that $\operatorname{Ind}(L') = -\operatorname{Ind}(L)$ for such a choice.

***Hint**: Vector subspaces of finite dimension or codimension admit complements.*

Exercise 2. *Let $u : E \longrightarrow F$ be a continuous linear map between Banach spaces, and let $L : E \longrightarrow F$ be Fredholm. Suppose that u has finite rank. Show that $L + u$ is Fredholm and that $\operatorname{Ind}(L + u) = \operatorname{Ind}(L)$.*

Use this to conclude that, for L' as in Exercise 1, we always have that $\operatorname{Ind}(L') = -\operatorname{Ind}(L)$.

Exercise 3. *Let $K : E \longrightarrow E$ be a compact operator on a Banach space. Show that $L = \mathbb{1}_E + K$ is Fredholm.*

***Hint**: To study the kernel of L, consider $\mathbb{1}_E|_{\operatorname{Ker} L}$. To study its cokernel, consider the dual map $L^* : E^* \longrightarrow E^*$.*

Exercise 4. *Let $L : E \longrightarrow F$ be linear and continuous between Banach spaces. Show that L is Fredholm if and only if there exists $L' : F \longrightarrow E$ linear and continuous such that $L \circ L' - \mathbb{1}_F$ and $L' \circ L - \mathbb{1}_E$ are compact operators.*

***Hint**: Finite-rank operators are compact.*

We usually call L' as above a *Fredholm inverse* of L.

Exercise 5 (Theorem 4.9.2)**.** *Let $L : E \longrightarrow F$ be a Fredholm operator, and let L' be as in Exercise 1. Show that, for every $u : E \longrightarrow F$ linear continuous with norm $||u|| < ||L'||^{-1}$, the operator $L + u$ is Fredholm and $\operatorname{Ind}(L + u) = \operatorname{Ind}(L)$.*

***Hint**: If $A : E \longrightarrow E$ is linear continuous with $||A|| < 1$, then $\mathbb{1}_E + A$ is invertible.*

Exercise 6. *Let $L : E \longrightarrow F$ be a Fredholm operator, and let $K : E \longrightarrow F$ be a compact one. Show that $L + K$ is Fredholm and that $\operatorname{Ind}(L + K) = \operatorname{Ind}(L)$. Conclude that, if L' is a Fredholm inverse of L, then $\operatorname{Ind}(L') = -\operatorname{Ind}(L)$.*