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Applied Stochastic Processes

Exercise sheet 1

Setup: All the random variables considered in the exercices are defined on a fixed underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Quiz 1.1 Which of the following matrices define transition probabilities?

1. $$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{pmatrix}.$$

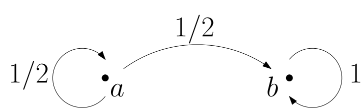
2. $$\begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix}.$$

3. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

5. $$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Quiz 1.2 Consider the Markov Chain X starting at a and with transition probability corresponding to the following weighted graph



Which of the following statements are true?

1. $\mathbb{P}(X_{10} = a, X_{100} = a) > 0$.
2. $\mathbb{P}(X_{10} = b, X_{100} = a) > 0$.
3. $\mathbb{P}(X_0 = a, X_1 = a, X_2 = a) = \mathbb{P}(X_0 = a, X_1 = a, X_2 = b)$.
4. For every $n \geq 1$ $p_{aa}^{(n)} = \mathbb{P}(X_0 = X_1 = \cdots = X_n)$.
5. For every $x \in \{a, b\}$, $\lim_{n\to\infty} p_{ax}^{(n)} = 0$.

Quiz 1.3 Let P, Q be two stochastic matrices. Which of the following are also stochastic matrices (for any choice of P and Q) ?

1. $P + Q$ (sum).
2. PQ (product).
3. P^t (transposition).
4. P^{-1} (inverse).
5. $\frac{1}{e}\exp(P)$ (rescaled exponential).
6. $\frac{1}{2}(P^{10} + Q^{11})$.

Exercise 1.4 [Markov chains] A die is rolled repeatedly. The outcome is a sequence of i.i.d. (independent, identically distributed) random variables $(\xi_n)_{n\geq 0}$ with $\mathbb{P}[\xi_0 = a] = 1/6$, $\forall a \in \{1, 2, \ldots, 6\}$. We consider the following sequences of random variables $X^j = (X_n^j)_{n\geq 0}$:

i) Let $X_n^1 = \xi_n$.

ii) Let $X_n^2 = \max\{\xi_0, \ldots, \xi_n\}$, the largest number that has come up in the first $n + 1$ rolls.

iii) Let $X_0^3 = \xi_0$ and $X_n^3 = \max\{\xi_n, \xi_{n-1}\}$ for $n \geq 1$, the larger number of those that came up in the rolls number $n - 1$ and n (the last two rolls).

iv) Let $X_n^4 = |\{\xi_0, \ldots, \xi_n\}|$, the number of different outcomes in the first $n + 1$ rolls.

v) Let X_n^5 denote the number of rolls at time n since the most recent six or since the start (whichever is smaller).

(a) Which sequence is a Markov chain $\mathrm{MC}(\mu, P)$ for some initial μ and some transition probability P?

For the options that you selected above:

(b) Determine μ and P.

(c) Represent P as a weighted oriented graph.

(d) Can you determine the n-step transition probability?

Exercise 1.5 [Deterministic Markov chains]

(a) Show that a deterministic sequence $(x_n)_{n\geq 0}$ is a Markov chain if and only if there exists a function $\Phi : S \to S$ such that for all $n \geq 0$,

$$x_{n+1} = \Phi(x_n).$$

(b) Determine *all* deterministic Markov chains on $S = \{1, 2, 3\}$.

Submission deadline: 10:15, Feburary 25.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`