ETH Zürich, FS 2025
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Mateo Rodriguez Polo

Applied Stochastic Processes

Exercise sheet 10

Quiz 10.1

(a) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be non-increasing and integrable. Let g be the solution of the (h, F) renewal equation. Which of the following are true?

1. $g = h + h \star m$.
2. $g + 10$ is also a solution of the (h, F) renewal equation.
3. $g + 10$ is the solution of the $(h + 10, F)$ renewal equation.
4. $g + 10 + 10F$ is a solution of the $(h + 10, F)$ renewal equation.
5. $g + m$ is a solution of the $(h + F, F)$ renewal equation.

(b) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be non-increasing and integrable. Let g be the solution of the (h, F) renewal equation. Which of the following are true?

1. $g(t)$ is eventually constant if and only if h is eventually zero.
2. For F lattice, g satisfies $\lim_{t\to\infty} g(t) = \frac{1}{\mu}\int_0^\infty h(u)du$.
3. For F non-lattice, g is the unique function that satisfies $\lim_{t\to\infty} g(t) = \frac{1}{\mu}\int_0^\infty h(u)du$ and solves the renewal equation.
4. If F is lattice with span $d > 0$, then $g(t)$ converges to some constant.
5. For F non-lattice, we have that $g(t + s) = h(t + s) + \int_0^{t+s} g(t + s - x)\, dF(x)$

Exercise 10.2 [$1 + m(t)$ **is subadditive**]
Prove that the function $u(t) = 1 + m(t)$ is subadditive. This is, prove that, for all $t, s \geq 0$, we have that $m(t + s) \leq 1 + m(t) + m(s)$.

Exercise 10.3 [**Age process - follow-up**]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F. Recall that the age process $(A_t)_{t\geq 0}$ of $(N_t)_{t\geq 0}$ is defined by

$$A_t = t - S_{N_t}.$$

For $x \geq 0$, we set $a_x(t) = \mathbb{P}[A_t \leq x]$ for $t \geq 0$. In Exercise 9.3, it was shown that a_x satisfies the renewal equation $a_x = h_x + a_x * F$, where $h_x(t) = \mathbb{1}_{\{t\leq x\}}(1 - F(t))$.

(a) Assume that F is non-lattice. Compute

$$\lim_{t\to\infty} a_x(t) \ .$$

Hint: Apply Smith's key renewal theorem.

(b) Deduce that A_t converges in distribution to some random variable A_∞ as $t \to \infty$.

Exercise 10.4 [**Cycles of operation and repair of a machine - follow-up**]
Let $(U_i, V_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i+V_i$

is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t = 1$ if the machine is operational at time t and $Y_t = 0$ otherwise. Let $g(t) = \mathbb{P}[Y_t = 1]$ denote the probability of the machine being operational at time $t \geq 0$, and $g(t) = 0$ for $t < 0$. We also define $h(t) = \mathbb{P}[U_1 > t]$. In Exercise 9.4, it was shown that for $t \geq 0$,

$$g(t) = h(t) + \int_0^t g(t-s) dF(s),$$

i.e. that g is the solution of the (h, F)-renewal equation.

(a) Assume that $\mathbb{E}[U_1] < \infty$ and that F is non-lattice. Show that the function h is non-increasing and integrable and conclude that

$$\lim_{t\to\infty} g(t) = \frac{\mathbb{E}[U_1]}{\mathbb{E}[U_1] + \mathbb{E}[V_1]}.$$

Exercise 10.5 [Poisson approximation]
Let $(p_n)_{n>0}$ be a sequence of parameters ($p_n \in [0,1]$) and $\lambda \in (0, \infty)$ such that

$$\lim_{n\to\infty} np_n = \lambda.$$

For every $n \geq 1$, let $X_n \sim \text{Bin}(n, p_n)$. Prove that X_n converges in distribution to a Poisson-distributed random variable X with parameter λ, i.e.

$$X_n \xrightarrow{(d)} X \sim \text{Pois}(\lambda) \quad \text{as } n \to \infty.$$

Hint: Consider $\mathbb{P}[X_n = k]$ for fixed $k \geq 0$ and show that it converges to $\frac{e^{-\lambda}}{k!}\lambda^k$.

Submission deadline: 10:15, May 6.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`