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Applied Stochastic Processes

Exercise sheet 11

Quiz 11.1

(a) Let U be a uniformly distributed random variable taking values in $[0,5]$. Which of the following statements are correct?

1. δ_U is a point process on $([0,5], \mathcal{B}([0,5]))$.
2. $2 \cdot \delta_U$ is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
3. δ_{2U} is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
4. U is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
5. $\frac{1}{2}\delta_U + \frac{1}{2}\delta_{-U}$ is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(b) Which of the following are correct?

1. If $X \sim \text{Poisson}(\lambda)$, then the characteristic function $\varphi_X(t) = \exp(\lambda(it-1))$ uniquely determines the distribution of X.
2. Let $X_n \sim \text{Poisson}(\lambda_n)$ with $\lambda_n \to \infty$. Then $\frac{X_n - \lambda_n}{\sqrt{\lambda_n}}$ converges in distribution to a standard normal random variable.
3. Consider a family of independent random variables with $X_n \sim \text{Poisson}(1/n)$. Then, the sequence X_n converges in probability to zero.
4. The conditional distribution of a Poisson variable $X \sim \text{Poisson}(\lambda)$, given $X > 0$, is still a Poisson distribution.
5. Let $M, N \sim \text{Poisson}(1)$ be two independent random variables. Then:

$$\mathbb{P}(M \text{ is odd, } N \text{ is odd}) = \mathbb{P}(M \text{ is even, } N \text{ is even, } M + N \neq 0)$$

Exercise 11.2 [Sums of independent Poisson random variables]

(a) Consider $k \geq 1$ and $\lambda_1, \ldots, \lambda_k \in [0, \infty)$. Let $X_i \sim \text{Pois}(\lambda_i)$, $1 \leq i \leq k$, be independent random variables. Show that

$$X_1 + \ldots + X_k \sim \text{Pois}(\lambda_1 + \ldots + \lambda_k).$$

(b) Consider $(\lambda_i)_{i \geq 1}$ with $\lambda_i \in [0, \infty]$, $i \geq 1$, and set $\lambda := \sum_{i=1}^{\infty} \lambda_i$. Let $X_i \sim \text{Pois}(\lambda_i)$, $i \geq 1$, be independent random variables. Show that

$$X := \sum_{i=1}^{\infty} X_i \sim \text{Pois}(\lambda).$$

Recall the convention that $X \sim \text{Pois}(\infty)$ if and only if $X = +\infty$ almost surely.

Exercise 11.3 [Measurable modification of a point process]
Let M be a point process on $(E, \mathcal{E})$. Recall that $M \in \mathcal{M}$ almost surely, where

$$\mathcal{M} = \{\text{sigma-finite measures } \eta \text{ on } E \text{ such that } \forall B \in \mathcal{E},\, \eta(B) \in \mathbb{N} \cup \{+\infty\}\}.$$

Let G be such that $G \subset \{M \in \mathcal{M}\}$ and $\mathbb{P}(G) = 1$, and define:

$$\widetilde{M}(\omega) = \begin{cases} M(\omega) & \text{if } \omega \in G, \\ 0 & \text{if } \omega \notin G. \end{cases}$$

Show that:

1. $\widetilde{M}$ is a random variable in $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{B}(\mathcal{M})$ is the sigma algebra generated by the sets of the form
$$C_{k,B} = \{\eta \in \mathcal{M} \,:\, \eta(B) = k\}, \quad k \in \mathbb{N},\, B \in \mathcal{E}.$$
2. Define P_M to be the law of $\widetilde{M}$ as a random variable in $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. Show that this law is the same for all choices of the set G.

Exercise 11.4 [Law of a point process]
Consider two point processes M and M' on $(E, \mathcal{E})$, and denote their laws by P_M respectively $P_{M'}$. Show that the following statements are equivalent:

(i) $P_M = P_{M'}$

(ii) For all $A \in \mathcal{B}(\mathcal{M})$,
$$\mathbb{P}[\widetilde{M} \in A] = \mathbb{P}[\tilde{M}' \in A].$$

(iii) For all $k \geq 1$, $B_1, \dots, B_k \in \mathcal{E}$, and $n_1, \dots, n_k \in \mathbb{N}$,
$$\mathbb{P}[M(B_1) = n_1, \dots, M(B_k) = n_k] = \mathbb{P}[M'(B_1) = n_1, \dots, M'(B_k) = n_k].$$

(iv) For all $k \geq 1$, $C_1, \dots, C_k \in \mathcal{E}$ disjoint, and $n_1, \dots, n_k \in \mathbb{N}$,
$$\mathbb{P}[M(C_1) = n_1, \dots, M(C_k) = n_k] = \mathbb{P}[M'(C_1) = n_1, \dots, M'(C_k) = n_k].$$

Submission deadline: 10:15, May 13.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`