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Applied Stochastic Processes

Exercise sheet 12

Quiz 12.1

(a) Which of the following are correct?

1. Let M be a Poisson point process on $\mathbb{R} \times [0,2]$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1$. $T\#M$ is a Poisson point process on $\mathbb{R}$.
2. Let M be a Poisson point process on $\mathbb{R}^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1$. $T\#M$ is a Poisson point process on $\mathbb{R}$.
3. Let M be a Poisson point process on $[0,2]^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1/2$. $T\#M$ is a Poisson point process on $[0,1]$.
4. Let M be a Poisson point process on $\mathbb{R}^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto (2x_2, 2x_1)$. $T\#M$ is a Poisson point process on $\mathbb{R}^2$.
5. Let M be a Poisson point process on $\mathbb{R} \times [0,\infty)$ with intensity measure $\mu = \text{Leb}$. Then the restricted process $M_{[0,1]^2}$, $M_{[0,2]^2}$, and $M_{[2,3]^2}$ are Poisson point processes.

(b) Let M be a Poisson point process on $(E, \mathcal{E})$ with (σ-finite) intensity measure μ and let M' be a Poisson point process on $(F, \mathcal{F})$ with (σ-finite) intensity measure ν. Define N via the product measure $N(\omega) := M(\omega) \otimes M'(\omega)$ on $(E \times F, \mathcal{E} \otimes \mathcal{F})$. Which of the following are correct?

1. N assigns positive mass to rectangles $A \times B$ if and only if $M(A) > 0$ and $M'(B) > 0$.
2. N is a σ-finite measure almost surely.
3. N is a point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$.
4. N is a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$.
5. If M and M' are independent, then N is a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$.

Exercise 12.2 [Simple Poisson point process I]
A measure ν on $(E, \mathcal{E})$ is *diffuse* if

$$\forall x \in E,\ \nu(\{x\}) = 0,$$

and we call it *simple* if

$$\forall x \in E,\ \nu(\{x\}) \leq 1.$$

Let M be a Poisson point process on $(E, \mathcal{E})$ with intensity measure μ. In this exercise, we show that

$$\mu \text{ is diffuse} \quad \Longrightarrow \quad \text{M is simple a.s..}$$

From now on, assume that μ is diffuse, and let $(E_i)_{i \in \mathbb{N}}$ be a partition of E such that each E_i is measurable and satisfies $\mu(E_i) < \infty$.

(a) Show that the restricted measures $\mu_1, \mu_2, \ldots$ are diffuse and that $M_{E_1}, M_{E_2}, \ldots$ are independent Poisson point processes with respective intensities $\mu_{E_1}, \mu_{E_2}, \ldots$.

(b) Show that for every $i \in \mathbb{N}$, M_{E_i} is almost surely simple.

Hint: Using the explicit construction of M_{E_i} from Proposition 7.10, prove that $\sum_{i=1}^{Z} \delta_{X_i}$ is almost surely simple.

(c) Deduce that M is almost surely simple.

Exercise 12.3 [Simple Poisson point process II]

(a) Let M be a Poisson point process on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mu = \text{Leb}$. Is the process almost surely simple?

(b) Let M be a Poisson point process on $\mathbb{R}$ with intensity measure μ given by $\mu(B) = |B \cap \mathbb{Z}|$ for $B \in \mathcal{B}(\mathbb{R})$. Is the process almost surely simple?

Exercise 12.4 [Mapping]
Let M be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \lambda \cdot \text{Leb}(\mathbb{R}^d)$, where $\lambda > 0$. Let B_r the ball of radius r around the origin.

(a) Consider the map $T : \mathbb{R}^d \to [0, \infty)$ defined by $x \mapsto ||x||_2 := \sqrt{x_1^2 + \ldots + x_d^2}$. Determine the intensity measure of the Poisson point process $T\#M$.

(b) Prove that a.s.

$$\lim_{r \to \infty} \frac{M(B_r)}{|B_r|} = \lambda$$

where $|B_r|$ is the volume of B_r.

Hint: Consider a sequence $(r_k)_{k \geq 0}$ with $|B_{r_k}| = k$ and study $M(B_{r_k} \setminus B_{r_{k-1}})$, the number of points in the annuli $(B_{r_k} \setminus B_{r_{k-1}})_{k \geq 1}$.

Exercise 12.5 [Poisson point processes on discrete spaces]
Let E be a finite or countable set, and let d be a metric on E such that the corresponding Borel σ-algebra is $\mathcal{P}(E)$ (the power set of E). Let μ be a measure on E such that $\mu(\{x\}) < \infty$ for every $x \in E$.

Prove that:

$$M \text{ is a ppp}(\mu) \iff \left(\exists (N_x)_{x \in E} \text{ independent with } N_x \sim \text{Poi}(\mu(\{x\})) \text{ such that } M = \sum_{x \in E} N_x \delta_x \right).$$

Submission deadline: 10:15, May 20.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`