ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 13

Quiz 13.1

(a) Let M be a Poisson point process on $(E, \mathcal{E})$ with (σ-finite) intensity measure μ.

1. Let ν be a probability measure on $(F, \mathcal{F})$. We can construct a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with intensity measure $\mu \otimes \nu$ by marking the process M.
2. Let ν be a finite measure on $(F, \mathcal{F})$. We can use a similar construction as in 1 to construct a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with intensity measure $\mu \otimes \nu$.
3. Let ν be an arbitrary measure on $(F, \mathcal{F})$. We can use a similar construction as in 1 to construct a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with intensity measure $\mu \otimes \nu$.
4. Let ν_1 and ν_2 be two different mark laws. The resulting marked processes on M are independent.
5. The base process M can be recovered from the marked process $\tilde{M}$.

(b) Let μ be a Probability measure on $\mathbb{R}$ Let $M = \sum_{i \leq N} \delta_{X_i}$ and $M' = \sum_{i \leq N'} \delta_{X'_i}$ be two independent Poisson processes with intensity μ. Which of the following are also Poisson point processes?

1. $M - M'$
2. $M^2 = (M^2(B))_{B \in \mathcal{B}(\mathbb{R})}$
3. $\sqrt{M} = (\sqrt{M}(B))_{B \in \mathcal{B}(\mathbb{R})}$
4. $M + M'$
5. $\sum_{i \leq N, j \leq N'} \delta_{(X_i, X'_j)}$

Exercise 13.2 [Accumulation Point]
Let M be a Poisson point process on $\mathbb{R}$ with Lebesgue intensity. Consider the map $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that $M' = T\#M$ is Poisson point process.

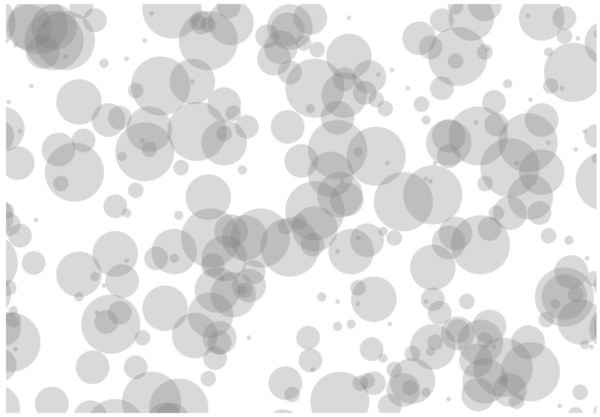
(b) Determine its intensity.

Exercise 13.3 [Poisson Boolean Percolation]
Let $M = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \mathrm{Leb}(\mathbb{R}^d)$. Let us consider $(R_i)_i$ a sequence of i.i.d. positive random variables with law ρ, and independent of M. We define the *occupied* set by $\mathcal{O} = \bigcup_i B(X_i, R_i)$, where $B(x, r) \subset \mathbb{R}^d$ is the open ball of center x and radius r.

(a) Let M_0 the number of balls $B(X_i, R_i)$ which contain the origin of $\mathbb{R}^d$. Show that M_0 is a well defined random variable with distribution Poisson $\left(\int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx)\right)$.

Hint: Use the marking theorem.

(b) Show that the event $\{\mathcal{O} = \mathbb{R}^d\}$ is measurable and that $\mathbb{P}[\mathcal{O} = \mathbb{R}^d] = 1$ if and only if $\int_0^\infty r^d \rho(dr) = \infty$.



Exercise 13.4 [Laplace functional]
Let M be a Poisson point process on $(E, \mathcal{E})$ with intensity measure μ. Recall that the Laplace functional $\mathcal{L}_M$ of N is given by

$$\mathcal{L}_M(u) = \mathbb{E}\left[\exp\left(-\int_E u(x)M(dx)\right)\right].$$

for every $u : E \to \mathbb{R}_+$ measurable.

(a) Let $B \in \mathcal{E}$. Show that if $\mu(B) < \infty$, then

$$\mu(B) = -\frac{d}{dt}\mathcal{L}_M(t1_B)\Big|_{t=0}.$$

(b) Let $B \in \mathcal{E}$. We no longer assume that $\mu(B) < \infty$. Show that

$$\mathbb{P}[M(B) = 0] = \lim_{t\to\infty} \mathcal{L}_M(t1_B).$$

Exercise 13.5 [Campbell's formula]
Let N be a Poisson point process on $(E, \mathcal{E})$ with intensity measure μ, and let $u : E \to \mathbb{R}$ be a measurable function. Show that $\int u(x)N(dx)$ is a well defined random variable and that if we have $u \geq 0$ or $\int |u(x)|\mu(dx) < \infty$, then

$$\mathbf{E}\left[\int u(x)N(dx)\right] = \int u(x)\mu(dx).$$

Submission deadline: 10:15, May 27.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`