ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 14

Quiz 14.1

(a) Let $(N_t)_{t\geq 0}$ be a counting process. Which of the following implies that N is a Poisson process with rate λ?

1. N is a Poisson point process with intensity measure $\lambda\,\mathrm{Leb}$ on $\mathbb{R}_+$.
2. N has independent and stationary increments.
3. N has independent and stationary increments and $N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s))$ for all $0 < s < t$.
4. N has the Markov property.
5. N is a renewal process with $T_1 \sim \mathrm{Exp}(\lambda)$.

(b) Which of the following are correct?

1. A stochastic process $(N_t)_{t\geq 0}$ with independent, stationary increments satisfying $N_0 = 0$ and $N_t \sim \mathrm{Pois}(t)$ is a Poisson process with rate 1.
2. A counting process with stationary increments has independent increments.
3. Let $(N_t)_{t\geq 0}$ be a counting process. Then $(N_t)_{t\geq 0}$ is almost surely càdlàg. [A function is *càdlàg* if it is right-continuous with left limits.]
4. Let $t \geq 0$. The number of jumps of $(N_t)_{t\geq 0}$ in the interval $[0,t]$ is almost surely finite.
5. For a Poisson process $(N_t)_{t\geq 0}$ with rate $\lambda > 0$, the waiting time S_n has a $\mathrm{Gamma}(n,\lambda)$ distribution.

Exercise 14.2 [Inhomogeneous Poisson process I]
Let $\rho : [0,\infty) \to (0,\infty)$ be a continuous function satisfying $\int_0^\infty \rho(u)du = \infty$. A counting process $(N_t)_{t\geq 0}$ is called an *inhomogeneous Poisson process* with rate ρ if it has independent increments and for all $t > s > 0$,

$$N_t - N_s \sim \mathrm{Pois}\left(\int_s^t \rho(u)du\right).$$

(a) For which choice of the function ρ do we obtain a Poisson process with rate λ?

(b) Does $(N_t)_{t\geq 0}$ have stationary increments?

Denote the jump times of $(N_t)_{t\geq 0}$ by $(S_i)_{i\geq 1}$, and consider the counting measure $M = \sum_{i\geq 1} \delta_{S_i}$. Analogously to the proof of Theorem 7.4, it is possible to prove that M is a Poisson point process on $\mathbb{R}_+$.

(c) What is the intensity measure μ_ρ of the Poisson point process M?

(d) Is S_1, the time until the first jump, independent of $S_2 - S_1$, the time between the first and the second jump?

Exercise 14.3 [Inhomogeneous Poisson process II]
Let $\rho : [0, \infty) \to (0, \infty)$ be a continuous function satisfying $\int_0^\infty \rho(u)du = \infty$.

(a) Define $R : [0, \infty) \to [0, \infty)$ by
$$R(t) = \int_0^t \rho(u)du.$$
Show that R is a continuous, increasing bijection.

(b) Let $(N_t)_{t\geq 0}$ be an inhomogeneous Poisson process with rate ρ. Show that $(\widetilde{N}_t)_{t\geq 0}$ defined by $\widetilde{N}_t := N_{R^{-1}(t)}$ is a Poisson process with rate 1.

(c) Let $(\widetilde{N}_t)_{t\geq 0}$ be a Poisson process with rate 1. Show that $(N_t)_{t\geq 0}$ defined by $N_t := \widetilde{N}_{R(t)}$ is an inhomogeneous Poisson process with rate ρ.

Exercise 14.4 [The Waiting Time Paradox]

(a) Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$. Let $(S_n)_{n\geq 1}$ be the jump times of the process. For a fixed $t > 0$, let $A_t = t - S_{N_t}$ be the time passed since the most recent jump (or after 0) in the process, and let $B_t = S_{N_t+1} - t$ be the time forward to the next jump. Let $T_1 \sim \text{Exp}(\lambda)$. Show that A_t and B_t are independent, that B_t is distributed as T_1 and that A_t is distributed as $T_1 \wedge t$.

(b) Let $L_t = A_t + B_t = S_{N_t+1} - S_{N_t}$ be the length of the inter-arrival interval covering t. Show that
$$\mathbf{E}[L_t] \to 2\mathbf{E}[T_1] \quad \text{as } t \to \infty.$$
Since L_t is the time between two consecutive jumps, we might expect $\mathbf{E}[L_t] = \mathbf{E}[T_1]$. Give an intuitive resolution of the apparent paradox.

Exercise 14.5 [Largest gap in a Poisson process]
Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. The largest gap up to time t is defined as
$$L_t = \max_{k\geq 1}(S_k \wedge t - S_{k-1} \wedge t).$$

(a) Let $\varepsilon > 0$. Show that there exists almost surely some $n_0 \geq 1$ such that for all $n \geq n_0$,
$$\max_{1\leq k\leq n} T_k \leq \frac{1+\varepsilon}{\lambda}\log(n/\lambda),$$
where the $T_1, \dots, T_n$ denote the inter-arrival times of the process.

Hint: Use Borel-Cantelli's lemma.

(b) Show that there exists almost surely some $t_0 \geq 0$ such that for all $t > t_0$,
$$N_t + 1 \leq (1+\varepsilon)t\lambda.$$

(c) Conclude that almost surely
$$\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}.$$

Submission deadline: 10:15, June 3.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`

The Applied Stochastic Processes team wishes you a great summer break and best of luck for your exam preparation!