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Applied Stochastic Processes

Exercise sheet 2

Quiz 2.1 [Framework for MC]
Let S be finite or countable and P be a transition probability.

(a) Let $x, y \in S$. Which of the following quantities are equal to p_{xy} (for arbitrary choices of P, x and y)?

1. $\mathbb{P}_x(X_1 = y, X_0 = x)$.
2. $\mathbb{P}_x(X_1 = y)$.
3. $\mathbb{P}_x(X_1 = x, X_2 = y)$.
4. $\mathbb{P}_x(X_1 = x, X_0 = y)$.
5. $\mathbb{P}_\mu(X_1 = y)$ with $\mu = \delta^x$.

(b) Let $n \geq 1$, $x, y, z \in S$. Which of the following expressions are equal to $\mathbb{P}_x(X_{n+2} = z, X_{n+1} = y, X_n = x)$ (for arbitrary choices of P, x, y, z and n)?

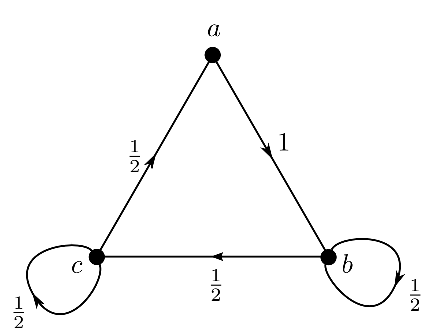
1. $\mathbb{P}_x(X_{n+2} = z, X_{n+1} = y | X_n = x)$.
2. $\mathbb{P}_x(X_{2n+2} = z, X_{2n+1} = y, X_{2n} = x | X_n = x)$.
3. $p_{xx}^{(n)} p_{xy} p_{yz}$.
4. $p_{xy} p_{yz}$.
5. $\sum_{u_1,\dots,u_{n-1} \in S} p_{xu_1} \cdot \ldots p_{u_{n-1}x} \cdot p_{xy} \cdot p_{yz}$.

Consider the SRW on $\mathbb{Z}$ with transition probability P given by $p_{ij} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$.

(c) Which of the following probabilities are 0?

1. $\mathbb{P}_1(X_2 = 3)$.
2. $\mathbb{P}_1(X_3 = 3)$.
3. $\mathbb{P}_1(X_4 = 3)$.
4. $\mathbb{P}_1(X_5 X_6 < 0)$.
5. $\mathbb{P}_1(X_{2n+1} = 0)$ for $n \in \mathbb{N}$.

Exercise 2.2 [n-step transition probability] Consider the three-state Markov chain with transition probability P given by the following diagram



Prove that

$$p_{a,a}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right).$$

Hint: What are the values of $p_{a,a}^{(0)}$, $p_{a,a}^{(1)}$, and $p_{a,a}^{(2)}$? Recall that $p_{a,a}^{(n)} = \delta^a P^n \delta^a$, where P is the representation of the transition probability as a 3×3 matrix. To find an expression for $\delta^a P^n \delta^a$, first compute the eigenvalues of P.

Exercise 2.3 [Complements I]

Let $(X_n)_{n\geq 0}$ be a sequence of random variables with values in S satisfying the 1-step Markov property and homogeneity. Show that there exist a distribution μ and a transition probability P such that

$$X \sim \mathrm{MC}(\mu, P).$$

Note: this establishes the converse of [**Proposition 1.2**], thereby showing that the 1-step Markov property and homogeneity characterize Markov chains.

Exercise 2.4 [Simple Markov property I]

Consider the SRW on $\mathbb{Z}$. Show that the two random variables

$$Z := \sum_{n=0}^{10} \mathbb{1}_{X_n=0} \quad \text{and} \quad Z' := \sum_{n=10}^{20} \mathbb{1}_{X_n = X_{10}}$$

have the same distribution and are independent under $\mathbb{P}_0$.

Exercise 2.5 [Simple Markov property II]

Consider the SRW on $\mathbb{Z}$. For $N \geq 0$, we define the hitting time $H_{-N,N} := \inf\{n \geq 0 : X_n \in \{-N, N\}\}$.

(a) Show that for every $k \geq 0$,

$$\mathbb{P}_0(H_{-N,N} > k \cdot N) \leq \left(1 - 2^{-N}\right)^k.$$

Deduce that $\mathrm{E}_0(H_{-N,N}) \leq N \cdot 2^N$.

(b) Show that $\mathrm{E}_x(H_{-N,N}) < \infty$ for all $x \in \{-N, \dots, N\}$.

(c) Prove that $\mathrm{E}_0(H_{-N,N}) = N^2$.

Hint: Consider the function $f(x) = \mathrm{E}_x(H_{-N,N})$ for $x \in \{-N, \dots, N\}$.

Submission deadline: 10:15, March 4.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`