ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 3

Quiz 3.1 [Quiz]

(a) Let $x, y, z \in S$, $n \geq 1$. Which of the following expression is equal to $\mathbf{P}_x(H_y < \infty, H_y > n \mid X_n = z)$? (for arbitrary x, y, z, n and Markov Chain)

1. $\mathbf{P}_z(H_y > n)\mathbf{P}_x(H_y < \infty \mid X_n = z)$.
2. $\mathbf{P}_x(H_y < \infty)\mathbf{P}_z(H_y > n \mid X_n = x)$.
3. $\mathbf{P}_z(H_y < \infty)\mathbf{P}_x(H_y > n \mid X_n = z)$.
4. $\mathbf{P}_x(H_y > n)\mathbf{P}_z(H_y < \infty \mid X_n = x)$.
5. $\mathbf{P}_x(H_y < \infty, H_y > 2n \mid X_{2n} = z)$.

(b) Let $x, y \in S$ be such that $x \neq y$. Let $T_1, T_2, \ldots$ be the inter-visit times at y. Which of the following statements hold? (for arbitrary x, y and Markov Chain)

1. $\mathbf{P}_x(T_1 = 2) = \mathbf{P}_x(T_2 = 2)$.
2. $\mathbf{P}_x(T_2 = 2) = \mathbf{P}_x(T_3 = 2)$.
3. $\mathbf{P}_y(T_1 = 2) = \mathbf{P}_y(T_2 = 2)$.
4. $X_{T_2} = y$ $\mathbf{P}_x$-almost surely.
5. $X_{T_1+T_2} = y$ $\mathbf{P}_x$-almost surely.

(c) In the same setup as the previous question, which of the following statements hold?

1. Under $\mathbf{P}_x$, T_2 and T_3 are always independent.
2. Under $\mathbf{P}_x$, T_1 and T_2 are always independent.
3. Under $\mathbf{P}_y$, T_1 and T_2 are always independent.
4. $\mathbf{P}_x(T_8 = \infty \mid T_7 < \infty) = \mathbf{P}_y(H_y = \infty)$.
5. $\mathbf{P}_x(T_8 = \infty \mid T_7 < \infty) = \mathbf{P}_x(H_x = \infty)$.

Exercise 3.2 [Alternative formulation of the Simple and Strong Markov Properties]
Let $f : S^{\mathbb{N}} \to \mathbb{R}$ be a measurable bounded function. Let $(X_n)_{n\geq 0}$ be a Markov Chain with initial distribution μ and $(\mathcal{F}_n)_{n\geq 0}$ its natural filtration. For every $x \in S$, define

$$g(x) = \mathbf{E}_x\left(f((X_n)_{n\geq 0})\right).$$

(a) Prove that

$$\mathbf{E}_\mu\left[f\big((X_{k+n})_{n\geq 0}\big) \mid \mathcal{F}_k\right] = g(X_k) \quad \mathbf{P}_\mu\text{-a.s.}$$

(b) Let T be a stopping time such that $\mathbf{P}(T < \infty) = 1$. Prove that

$$\mathbf{E}_\mu\left[f\big((X_{T+n})_{n\geq 0}\big) \mid \mathcal{F}_T\right] = g(X_T) \quad \mathbf{P}_\mu\text{-a.s.}$$

Exercise 3.3 [Hitting times for the SRW on $\mathbb{Z}$]
Consider the simple random walk (SRW) on $\mathbb{Z}$, i.e., the Markov chain with state space $S = \mathbb{Z}$ and transition probabilities $p_{xy} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$. Define, for every $k \in \mathbb{Z}$,

$$\widetilde{H}_k = \min\{n \geq 0 : X_n = k\}.$$

Fix $N \in \mathbb{N} \setminus \{0\}$, and define the function $h : \{0, 1, \ldots, N\} \to [0, 1]$ by

$$h(x) = \mathbf{P}_x(\widetilde{H}_N < \widetilde{H}_0).$$

(a) Prove that h satisfies the following conditions:

$$\begin{aligned} &h(0) = 0, \quad h(N) = 1, \\ &h(x) = \frac{1}{2}h(x+1) + \frac{1}{2}h(x-1), \quad \forall x \in \{1, \ldots, N-1\}. \end{aligned}$$

(b) Deduce that for all $x \in \{0, \ldots, N\}$,

$$h(x) = \frac{x}{N}.$$

Exercise 3.4 [Exponential tail of exit time from a finite set]
Comment: This exercise is a generalization of the result in Exercise 2.5 (a).
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities $(p_{x,y})_{x,y\in S}$. Let $C \subseteq S$ such that $S \setminus C$ is finite. Define $n(x) := \min\{n \geq 0 : \mathrm{P}_x(X_n \in C) > 0\}$, and suppose that $n(x) < \infty$ for all $x \in S$. Let

$$\begin{aligned} \tau_C &= \inf\{n \geq 0 : X_n \in C\}, \\ \varepsilon &= \min\{\mathbf{P}_x(X_{n(x)} \in C) : x \in S\}, \\ N &= \max\{n(x) : x \in S\}. \end{aligned}$$

Show that for all $k \in \mathbb{N}$ and for every $x \in S$,

$$\mathbf{P}_x(\tau_C > kN) \leq (1 - \varepsilon)^k.$$

Exercise 3.5 [Reflection principle for the SRW on $\mathbb{Z}$]
Consider the SRW on $\mathbb{Z}$, i.e. the Markov chain with transition probability given by $p_{ij} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$. The goal of this exercise is to prove that for $n \geq 0$ even and $a \geq 1$ odd,

$$\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbf{P}_0\big(|X_n| \geq a\big). \tag{1}$$

(a) Show that

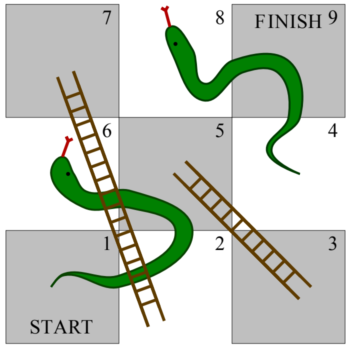
$$\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbf{P}_0(X_n > a) + \mathbf{P}_0(H_a \leq n, X_n < a).$$

(b) Use the strong Markov property to show that

$$\mathbf{P}_0(H_a \leq n, X_n < a) = \mathbf{P}_0(X_n > a),$$

and conclude that (1) holds.

Exercise 3.6 [Snakes and ladders]
A simple game of 'snakes and ladders' is played on a board of nine squares.



At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail.

(a) How many turns on average does it take to complete the game?
Hint: Call $k_i = \mathbf{E}_i(H_9)$ and find some relations between the k_i for $i \in \{1, \ldots, 9\}$.

(b) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Submission deadline: 10:15, March 11.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`