ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 4

Quiz 4.1 [Quiz]
Let $x, y \in S$. For $x \in S$, we recall the definitions $H_x = \min\{n \geq 1 : X_n = x\}$,

$$V_x = \sum_{k \geq 1} \mathbf{1}_{X_k = x}, \quad \text{and} \quad V_x^{(n)} = \sum_{k=1}^{n} \mathbf{1}_{X_k = x}.$$

In addition, we define $\widetilde{H_x} = \min\{n \geq 0 : X_n = x\}$, and

$$\widetilde{V_x} = \sum_{k \geq 0} \mathbf{1}_{X_k = x}.$$

(a) Which of the following statements are correct?

1. $|H_x - \widetilde{H_x}| \leq 1$ a.s.
2. $\mathbf{P}_y[\widetilde{H_x} < \infty] = \mathbf{P}_y[H_x < \infty]$ for all x, y.
3. $\mathbf{P}_y[\widetilde{H_x} < \infty] = \mathbf{P}_y[H_x < \infty]$ for all $x \neq y$.
4. $\mathbf{E}_x[\widetilde{V_x}] = 1 + \mathbf{E}_x[V_x]$ for all x.
5. $\widetilde{V_x} = V_x \quad \mathbb{P}_y$−a.s for all $x \neq y$.

(b) Now let us fix $x \neq y$. Which of the following statements are correct?

1. $\mathbf{E}_y[V_x] = \mathbf{P}_y[H_x < \infty] \cdot (1 + \mathbf{E}_x[V_x])$.
2. $\mathbf{E}_y[V_x^{(n)}] = \mathbf{E}_y[V_x^{(n+1)}]$ for all $n \geq 1$.
3. $\lim_{n\to\infty} \mathbf{E}_y[V_x^{(n)}] = \mathbf{E}_y[V_x]$.
4. It is possible that $\mathbf{P}_x[V_x = \infty] = 1/2$.
5. It is possible that $\mathbf{P}_x[V_x = 2] = 1/8$.

Exercise 4.2 [Biased and reflected random walk]
Let $\alpha \in (0, 1)$. We consider the biased random walk X on $\mathbb{Z}$, i.e. the Markov chain with state space $\mathbb{Z}$ and transition probability given by

$$p_{x,x+1} = \alpha, \quad \text{and} \quad p_{x,x-1} = 1 - \alpha \quad \text{for } x \in \mathbb{Z}.$$

(a) Let $x \in \mathbb{Z}$. Show that x is recurrent if $\alpha = 1/2$.

Reminder: If $Z_1, Z_2, \dots$ are iid uniform in $\{-1, +1\}$ then there exists a constant $c > 0$ such that $\mathbb{P}(Z_1 + \cdots + Z_{2n} = 0) \sim \frac{c}{\sqrt{n}}$ as $n \to \infty$.

(b) Let $x \in \mathbb{Z}$. Show that x is transient if $\alpha \neq 1/2$.

Hint: Use the strong law of large numbers.

We now consider the reflected random walk Y on $\mathbb{N}$, i.e. the Markov chain with state space $\mathbb{N}$ and transition probability given by $p_{0,1} = 1$,

$$p_{x,x+1} = \alpha, \quad \text{and} \quad p_{x,x-1} = 1 - \alpha \quad \text{for } x \geq 1.$$

(c) Show that 0 is recurrent if $\alpha \leq 1/2$.

Hint: Construct a coupling with the biased random walk X.

(d) Show that 0 is positive recurrent if $\alpha < 1/2$.

(e) Show that 0 is transient if $\alpha > 1/2$.

As a direct consequence of the Classification of States Theorem (next week), it will follow that any state x is recurrent if and only if $0 < \alpha \leq 1/2$.

Exercise 4.3 [SRW on a d-regular tree]
Let $d \geq 3$. In this exercise, we consider the simple random walk X on the d-regular tree $\mathbf{T}_d = (V, E)$, i.e. the infinite tree with d edges at each vertex. This is the Markov chain with state space V and transition probability given by

$$p_{xy} = \frac{1}{d} \cdot \mathbf{1}_{\{x,y\} \in E} \ .$$

Show that every state $x \in V$ is transient.
Hint: Under $\mathbf{P}_x$, consider the distance of X_n from the starting point x.

Exercise 4.4 [Returning Markov Chain]
Let μ be a measure on $\mathbb{Z}^2$. Consider the Markov Chain X with the following transition probability for $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$p_{xy} = \begin{cases} \mu(y), & \text{if } x_1 = x_2 = 0, \\ 1, & \text{if } x_2 \neq 0,\ x_1 = y_1 \text{ and } x_2 - \operatorname{sign}(x_2) = y_2, \\ 1, & \text{if } x_2 = 0,\ x_1 \neq 0,\ x_2 = y_2 \text{ and } x_1 - \operatorname{sign}(x_1) = y_1, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, from the state $(0,0)$ the Markov Chain jumps according to μ, and then it deterministically returns to the origin, first moving vertically and then horizontally.

(a) Prove that $(0,0)$ is recurrent.

(b) Study the positive or null recurrence of $(0,0)$.

(c) Prove or disprove that the following are Markov Chains.

1. $(\|X_n\|_\infty)_{n\geq 0}$.
2. $(\|X_n\|_1)_{n\geq 0}$.
3. $(\Pi_x(X_n))_{n\geq 0}$. Here, $\Pi_x(x, y) = x$.

Submission deadline: 10:15, March 18.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`