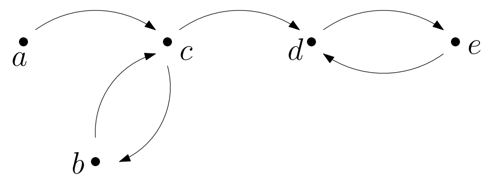
Ritvik Radhakrishnan
Mateo Rodriguez Polo

Applied Stochastic Processes

Exercise sheet 5

Quiz 5.1 [Quiz]

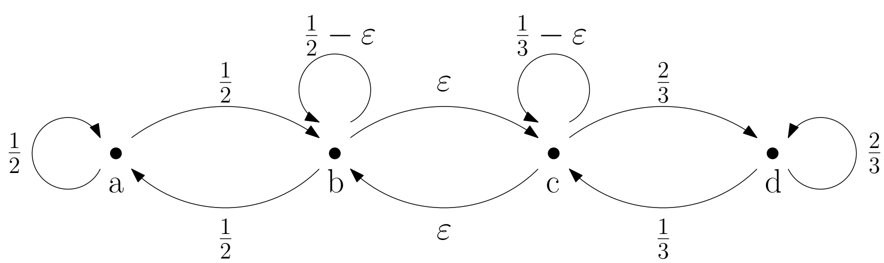
(a) Consider the transition probability represented by the diagram below.



Which of the following statements are correct?

1. In the chain above, there are two communicating classes.
2. In the chain above, the class to which b belongs is recurrent.
3. In the chain above, the class to which d belongs is recurrent.
4. There exists an *irreducible* chain with infinitely many recurrent classes.
5. There exists a chain with infinitely many recurrent classes.

(b) For $0 \leq \varepsilon \leq \frac{1}{3}$, consider the Markov chain on $S = \{a, b, c, d\}$ with transition probability P given by the following diagram.



Which of the following statements are correct?

1. For all $\varepsilon \in [0, 1/3]$, P is irreducible.
2. There exists $\varepsilon \in [0, 1/3]$ such that $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is reversible.
3. There exists $\varepsilon \in [0, 1/3]$ such that $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is reversible.
4. There exists $\varepsilon \in [0, 1/3]$ such that $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is stationary.
5. There exists $\varepsilon \in [0, 1/3]$ such that $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ is stationary.

Exercise 5.2 [Galton-Watson chain]
Let ν be a probability measure on $\mathbb{N}$ and let $(Z_i^n)_{i,n\geq 1}$ be independent ν-distributed random variables with $\nu(0), \nu(1) > 0$ and $\nu(0) + \nu(1) < 1$. The Galton-Watson chain $(X_n)_{n\geq 0}$ is defined by $X_0 = 1$ and for $n \geq 0$ by

$$X_{n+1} = \begin{cases} Z_1^{n+1} + Z_2^{n+1} + \ldots + Z_{X_n}^{n+1} & \text{if } X_n > 0, \\ 0 & \text{if } X_n = 0. \end{cases}$$

(a) Show that $(X_n)_{n\geq 0}$ is a Markov chain $\mathrm{MC}(\delta^1, P)$ on $\mathbb{N}$, and express P in terms of ν.

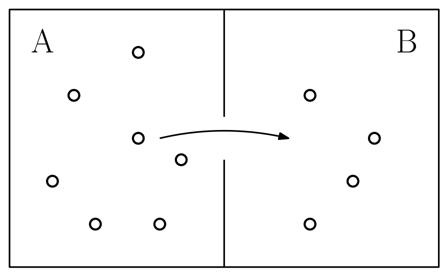
(b) Identify all communication classes. Which communication classes are closed?

(c) Which communication classes are recurrent? Which are transient?

(d) How do your answers to (b) and (c) change if $\nu(0) = 0$ and $\nu(1) < 1$?

Exercise 5.3 [Gas in Containers (Ehrenfest Model)]

Idea: Imagine two containers A and B with gas particles, and a small hole between them through which the particles can pass. At every step, a single particle is selected uniformly at random and passes through this hole.
Mathematical model: Let N be the total number of gas particles, and let X_n be the number of particles in A at time n. We want to model $(X_n)_{n\geq 0}$ as a Markov chain.



(a) Determine a suitable state space S and transition probability P.

(b) Identify a stationary distribution π for P.

Hint: Try to find a reversible distribution for P.

Exercise 5.4 [Two-state Markov chain]
Let $p, q \in [0, 1]$. Consider the two state Markov chain on $S = \{1, 2\}$ with transition probability P given by

$$p_{11} = p, \quad p_{12} = 1 - p, \quad p_{22} = q, \quad \text{and} \quad p_{21} = 1 - q.$$

Prove that a distribution π is stationary for P if and only if it is reversible for P.

Exercise 5.5 [Time-reversed Markov chain]
Let S be finite or countable. Consider a transition probability $P = (p_{xy})_{x,y\in S}$ and a stationary distribution π.

(a) Define $\hat{P} = (\hat{p}_{xy})_{x,y\in S}$ by

$$\hat{p}_{xy} = \begin{cases} \frac{\pi_y p_{yx}}{\pi_x} & \text{if } \pi_x > 0, \\ \mathbb{1}_{x=y} & \text{if } \pi_x = 0. \end{cases}$$

Show that $\hat{P}$ is a transition probability. $\mathrm{MC}(\pi, \hat{P})$ is called the *time-reversal* of $\mathrm{MC}(\pi, P)$.

(b) Recall that a transition probability P can be represented by a linear operator $P : \mathrm{L}^\infty(S) \to \mathrm{L}^\infty(S)$, defined by

$$(Pf)(x) = \sum_{y\in S} p_{xy} f(y).$$

Define an inner product $\langle \cdot, \cdot \rangle_\pi$ on $\mathrm{L}^\infty(S)$ by

$$\langle f, g \rangle_\pi := \sum_{x \in S} f(x) g(x) \pi_x.$$

Show that the operators P and $\hat{P}$ are adjoint, i.e. $\langle Pf, g \rangle_\pi = \langle f, \hat{P} g \rangle_\pi$ for every $f, g \in \mathrm{L}^\infty(S)$.

(c) Show that the operator P is self-adjoint if and only if π is reversible (for P).

Submission deadline: 10:15, March 25.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`