ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

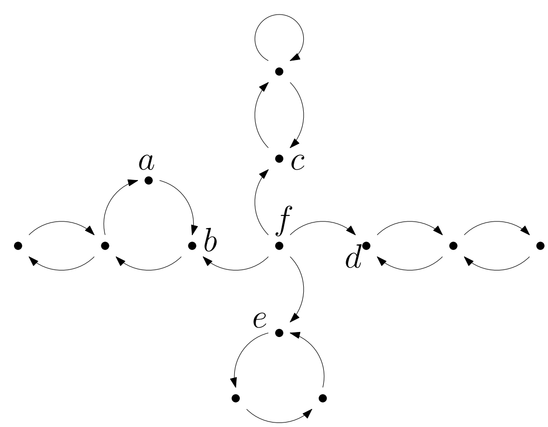
Exercise sheet 6

Quiz 6.1

(a) Let S be finite or countable. Let P be irreducible. Which of the following are true?

1. It is possible to choose S, P and $x \in S$ such that
$$\sum_{y \in S} \frac{1}{\mathbf{E}_x[H_y]} = 2.$$
2. It is possible to choose S, P such that
$$\sum_{y \in S} \frac{1}{\mathbf{E}_y[H_y]} = 2.$$
3. It is possible to choose S, P such that
$$\sum_{y \in S} \frac{1}{\mathbf{E}_y[H_y]} = 1.$$
4. If S is finite, there exists a unique stationary distribution.
5. If there exists a reversible distribution, it is unique.

(b) Consider the transition pobability represented by the following directed graph. Which of the following are true?



1. The period of a is 3.
2. The period of b is 1.
3. The period of c is 2.
4. The period of e is 3.
5. The period of f is 0.

Exercise 6.2 [Biased random walk on $\{1,\dots,N\}$]
Let $\alpha \in [0,1]$. Consider the biased random walk on $S = \{1,\dots,N\}$ with transition probability given by

$$p_{ij} = \begin{cases} \alpha & \text{if } j = i+1 \,(\operatorname{mod} N), \\ 1-\alpha & \text{if } j = i-1 \,(\operatorname{mod} N). \end{cases}$$

(a) Show that π, defined by $\pi(i) = 1/N, \forall i \in \{1,\dots,N\}$, is the unique stationary distribution for the biased random walk.

(b) Show that the biased random walk is reversible if and only if $\alpha = 1/2$.

Exercise 6.3 [Stationary measures]
Consider a chain P with state space S (finite or countable). Fix $u \in S$ positive recurrent and set $T = H_u$. Define the measure ν by

$$\forall x \in S \qquad \nu_x := \mathbf{E}_u\left[\sum_{n=1}^{T} \mathbf{1}_{X_n = x}\right].$$

(a) Show that $\frac{1}{m_u}\nu$ is a *stationary distribution.*

(b) Let x be in the same communication class as u. Show that

$$\nu_x = \frac{\mathbf{P}_u[H_x < H_u]}{\mathbf{P}_x[H_u < H_x]}.$$

(c) Assume, additionally, that P is irreducible, show that

$$\frac{m_u}{m_x} = \frac{\mathbf{P}_u[H_x < H_u]}{\mathbf{P}_x[H_u < H_x]}.$$

(d) Does this result still hold without the assumption that P is irreducible?

Exercise 6.4 [Lazy Markov chain]
Let $\delta \in (0,1)$, and let P be an irreducible transition probability. Define for all $x, y \in S$,

$$\widetilde{p}_{xy} = \delta \cdot \mathbb{1}_{x=y} + (1-\delta)\cdot p_{xy}.$$

(a) Show that $\widetilde{P} = (\widetilde{p}_{xy})_{x,y\in S}$ is a transition probability.

(b) Prove that $\widetilde{P}$ is irreducible and aperiodic.

(c) Assume that P positive recurrent. Prove that $\widetilde{P}$ is positive recurrent. What is the stationary distribution for $\widetilde{P}$?

Submission deadline: 10:15, April 1.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`