ETH Zürich, FS 2025
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Mateo Rodriguez Polo

Applied Stochastic Processes

Exercise sheet 7

Quiz 7.1

(a) Let P be the transition probability of the simple random walk on $\mathbb{Z}$ defined by $p_{ij} = \frac{1}{2}\mathbf{1}_{|i-j|=1}$ for every $i, j \in \mathbb{Z}$.

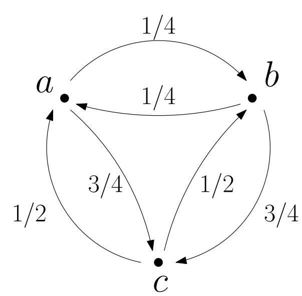
1. P is aperiodic.
2. P is irreducible.
3. P^2 is irreducible.
4. 0 is transient for P^2.
5. $\lim_{n\to\infty} \mathbf{P}_0[X_n = 0] = 0$.

(b) Consider an irreducible transition probability P on $S = \{1, ...10\}$. Which of the following statement hold.

1. It is possible that $\liminf \mathbf{P}_1(X_n = 1) = 0$.
2. It is possible that $\limsup \mathbf{P}_1(X_n = 1) = 0$.
3. There always exists i such that $\limsup \mathbf{P}_1(X_n = i) \geq \frac{1}{10}$.
4. $(P \text{ aperiodic}) \iff (\liminf \mathbf{P}_1(X_n = 1) = \limsup \mathbf{P}_1(X_n = 1))$.
5. $(P \text{ aperiodic}) \iff (\forall i \liminf \mathbf{P}_1(X_n = i) = \limsup \mathbf{P}_1(X_n = i))$.

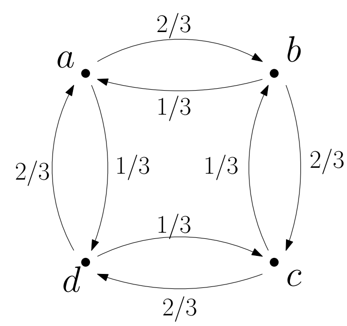
Exercise 7.2 [Convergence to equilibrium I]
Consider the three-state Markov chain X represented by the following directed graph. What is

$$\lim_{n\to\infty} \mathbf{P}_b[X_n = b] \ ?$$



Exercise 7.3 [Convergence to equilibrium II]
Consider the four-state Markov chain $X = (X_n)_{n\geq 0}$ represented by the following directed graph. What is

$$\lim_{n\to\infty} \mathbf{P}_c[X_{2n} = a] \ ?$$



Hint: Determine the transition probability of the Markov chain $(X_{2n})_{n\geq 0}$ with $X_0 = c$.

Exercise 7.4 [Hardcore model I]
Recall the hardcore model from Section 4.9. In this exercise, we consider the hardcore model on a 2x2 square grid. Denoting the vertices by $V = \{a, b, c, d\}$, this is the graph $G = (V, E)$ with edge set

$$E = \Big\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\Big\}.$$

(a) Identify all admissible configurations ξ. How many are there?

(b) Determine the transition probability P and represent it as a directed graph.

(c) Deduce that P is aperiodic and irreducible.

Exercise 7.5 [Hardcore model II]
Recall the hardcore model from Section 4.9. This is the Markov chain $X = (X_n)_{n\geq 0}$ on

$$S = \{\xi \in \{0, 1\}^V : \xi \text{ is admissible}\}$$

with transition probability P as defined in the proof of Proposition 4.17.

(a) Show that P is aperiodic.

(b) Show that P is irreducible.

In Section 4.9, this Markov chain was used to simulate a uniform random variable in S. Now, we fix $0 \leq k \leq 32$, and consider the set of admissible configurations with exactly k particles

$$S_k := \Big\{\xi \in S : \sum_{v\in V} \xi(v) = k\Big\}.$$

(c) How could we simulate Z, a uniform random variable in S_k?

Submission deadline: 10:15, April 8.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`