ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 8

Quiz 8.1

(a) Let (N_t) and (N'_t) be two independent renewal processes. Which of the following statements are true?

1. $(3N_t)$ is also a renewal process.
2. We always have $\lim_{t\to\infty} \frac{N_t}{t^2} = 0$ almost surely.
3. We always have $\lim_{t\to\infty} \frac{N_t + N'_t}{t} = 0$ almost surely.
4. $(N_t - N'_t)$ is also a renewal process.
5. $(N_t + N'_t)$ is also a renewal process.

(b) Let $\{T_n\}_{n\geq 1}$ be i.i.d. random variables with distribution $T_1 \sim \mathcal{U}(0, 2)$, and define their corresponding renewal process $(N_t)_{t\geq 0}$. Which of the following statements are true?

1. $(N_t)_{t\geq 0}$ has independent increments[1].
2. $(N_t)_{t\geq 0}$ has stationary increments[2].
3. The process $(N_n)_{n\geq 0}$ is a Markov process.
4. N_n is Uniform on $\{0, ..., 2n\}$.
5. $\lim_{t\to\infty} \frac{\mathbf{E}(N_t)}{t} = 1$.

Exercise 8.2 [Renewal function]
For $(N_t)_{t\geq 0}$, a renewal process with arrival distribution F, we define the *renewal function* $m : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$t \mapsto m(t) := \mathbf{E}(N_t).$$

Draw m for the following distributions of the inter-arrival times:

(a) $T_1 = 1$ a.s.

(b) $T_1 \sim \mathcal{U}(0, 1)$

(c) $T_1 \sim \text{Exp}(2)$

(d) $T_1 \sim \text{Ber}(1/2)$

[1] A stochastic process $(N_t)_{t\geq 0}$ has independent increments if, for any sequence of times $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $N_{t_2} - N_{t_1},\ N_{t_3} - N_{t_2},\ \ldots,\ N_{t_n} - N_{t_{n-1}}$ are independent.

[2] A stochastic process $(N_t)_{t\geq 0}$ has stationary increments if, for all $s, t \geq 0$, the increment $N_{s+t} - N_s$ has the same distribution as N_t. That is, the law of the increment depends only on the length of the time interval, not on its position.

Exercise 8.3 [CLT for renewal processes]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F and denote $\mu = E[T_1]$. Assume that $\mathbf{E}[{T_1}^2] < \infty$ and $\sigma^2 := \mathrm{Var}(T_1) > 0$. Show that

$$\frac{N_t - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{\text{law}} \mathcal{N}(0,1) \quad \text{as } t \to \infty,$$

where $\mathcal{N}(0,1)$ is the standard normal distribution.
Hint: Let $S_n := T_1 + \dots + T_n$, then by the central limit theorem

$$\lim_{n\to\infty} \mathbf{P}\Big((S_n - n\mu)/\sigma\sqrt{n} \leq x\Big) = \Phi(x)$$

uniformly in $x \in \mathbb{R}$, where Φ denotes the distribution function of the standard normal distribution.

Exercise 8.4 [Renewal process with lattice arrival distribution]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution given by T_1, and assume that the law of T_1 is lattice with span a. Denote $\mu := \mathbf{E}(T_1)$.

(a) Show that the span

$$a := \max\{a' > 0 : \mathbf{P}(T_1 \in a'\mathbb{Z}) = 1\}$$

is well-defined and finite.

(b) Show that $(\widetilde{N}_t)_{t\geq 0}$, defined by $\widetilde{N}_t := N_{at}$, is a renewal process with integer-valued jump times.

From now on, we assume that T_1 is lattice with span 1 and $\mathbf{P}(T_1 = 0) = 0$. We define

$$S := \begin{cases} \{0, 1, \dots, N-1\} & \text{if } N := \sup\{n \geq 0 : \mathbf{P}(T_1 = n) > 0\} < \infty, \\ \mathbb{N} & \text{otherwise,} \end{cases}$$

and for $i \geq 1$,

$$p_{0,i-1} = \mathbf{P}(T_1 = i), \quad \text{and} \quad p_{i,i-1} = 1.$$

(c) Show that $p = (p_{ij})_{i,j\in S}$ is a transition probability, and that the chain P is aperiodic, irreducible, and recurrent.

As in Exercise 7.2, we define the *renewal function* $m : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$t \mapsto m(t) := \mathbf{E}(N_t).$$

(d) *[Elementary renewal theorem]:* Show that

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Hint: Use the theorem “density of visit times” for Markov chains from Section 2.6.

(e) *[Blackwell's renewal theorem]:* Show that for all $k \in \mathbb{N}$,

$$\lim_{t\to\infty} m(t+k) - m(t) = \frac{k}{\mu}.$$

Hint: Use the results on the convergence of Markov chains from Sections 3.7 and 3.8.

Submission deadline: 10:15, April 15.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`