ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Exercise sheet 9

Quiz 9.1

(a) Let T_i, $i \geq 1$, be iid Uniform random variables in $[0, 3]$. In which case does $(N_t)_{t\geq 0}$ define a renewal process?

1. $N_t = \sum_{i\geq 1} \mathbf{1}_{T_1^2+\cdots+T_i^2 \leq t}$.
2. $N_t = \sum_{i\geq 1} \mathbf{1}_{T_2+\cdots+T_{i+1} \leq t}$.
3. $N_t = \sum_{i\geq 1} \mathbf{1}_{T_1^2+\cdots+T_i^2 \leq t}$.
4. $N_t = \sum_{i\geq 1} \mathbf{1}_{i \leq t}$.
5. $N_t = \sum_{i\geq 1} \mathbf{1}_{i^2 \leq t}$.

(b) Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution given by T_1 and denote $\mu = \mathbb{E}[T_1]$. Recall that $m(t) := \mathbb{E}[N_t]$ for $t \geq 0$. Which of the following are true?

1. If $T_1 \sim \text{Exp}(\lambda)$ for $\lambda > 0$, then we have that $m(t) = \frac{t}{\mu}$ for all $t \geq 0$.
2. If $T_1 \sim \text{Uniform}([0, 1])$, then we have that $m(t) = \frac{t}{\mu}$ for all $t \geq 0$.
3. $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$.
4. If the arrival distribution is non-lattice, $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$.
5. If the arrival distribution is lattice, $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$.

Exercise 9.2 [Convolution operator]
Let $G_1, G_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be right-continuous, non-decreasing functions.

(a) Show that $G_1 * G_2$ is right-continuous and non-decreasing and that $G_1 * G_2 = G_2 * G_1$.

(b) Let $h : \mathbb{R}^+ \to \mathbb{R}$ s.t. the convolutions are well-defined. Show that $(h * G_1) * G_2 = h * (G_1 * G_2)$.

Exercise 9.3 [Age process]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F. Denote by $(A_t)_{t\geq 0}$ the age process of $(N_t)_{t\geq 0}$, defined by

$$A_t = t - S_{N_t}.$$

For $x \geq 0$, set $a_x(t) = \mathbb{P}[A_t \leq x]$ for $t \geq 0$. Show that a_x satisfies the renewal equation

$$a_x(t) = \mathbf{1}_{t\leq x}(1 - F(t)) + \int_0^t a_x(t - s)dF(s) \quad \text{for } t \geq 0,$$

i.e. $a_x = h_x + a_x * F$, where $h_x(t) = \mathbb{1}_{\{t\leq x\}}(1 - F(t))$.

Exercise 9.4 [Cycles of operation and repair of a machine]
Let $(U_i, V_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i + V_i$ is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t = 1$ if the machine is operational at time t and $Y_t = 0$ otherwise. Let $g(t) = \mathbb{P}\left[Y_t = 1\right]$ denote the probability of the machine being operational at time $t \geq 0$, and $g(t) = 0$ for $t < 0$. We also define $h(t) = \mathbb{P}[U_1 > t]$. Prove that for $t \geq 0$

$$g(t) = h(t) + \int_0^t g(t-s)dF(s),$$

i.e. that g is the solution of the (h, F)-renewal equation.

Submission deadline: 10:15, April 29.

Please submit your solutions online before the beginning of the lecture.
Further information is available on:
`https://metaphor.ethz.ch/x/2025/fs/401-3602-00L/`