ETH Zürich, FS 2025
D-MATH
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Applied Stochastic Processes

Solution sheet 1

Solution 1.1

For the matrix $P = (p_{ij})_{i,j\in\{1,2,3\}}$ to define a transition probability, the two following properties must hold:

1. $p_{ij} \geq 0$, for all $i, j \in \{1, 2, 3\}$.
2. $\sum_{j\in S} p_{ij} = 1$, for all i.

The matrix 1 violates the second condition, while the matrix 5 violates the first condition. The remaining matrices define transition probabilities and correspond to Markov Chains on $S = \{1, 2, 3\}$ according to Theorem 1.3 of the Lecture Notes.

Solution 1.2

1. $\mathbb{P}(X_{10} = a, X_{100} = a) > 0$.

   True. We have:

$$\mathbb{P}(X_{10} = a, X_{100} = a) \geq \mathbb{P}(X_n = a, n = 0, ..., 100) = \frac{1}{2^{100}} > 0.$$

2. $\mathbb{P}(X_{10} = b, X_{100} = a) > 0$.

   False. Once the chain reaches b, it remains there forever because $p_{bb} = 1$. More formally,

$$\mathbb{P}(X_{100} = b|X_{10} = b) = 1 \implies \mathbb{P}(X_{10} = b, X_{100} = b) = \mathbb{P}(X_{10} = b),$$

   and so $\mathbb{P}(X_{10} = b, X_{100} = a) = 0$.

3. $\mathbb{P}(X_0 = a, X_1 = a, X_2 = a) = \mathbb{P}(X_0 = a, X_1 = a, X_2 = b)$.

   True. We compute:

$$\mathbb{P}(X_0 = a, X_1 = a, X_2 = a) = \mathbb{P}(X_0 = a)p_{aa}p_{aa} = 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

   Similarly,

$$\mathbb{P}(X_0 = a, X_1 = a, X_2 = b) = \mathbb{P}(X_0 = a)p_{aa}p_{ab} = 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

   Since both are equal, the statement is **true**.

4. For every $n \geq 1$, $p_{aa}^{(n)} = \mathbb{P}(X_0 = X_1 = \cdots = X_n)$.

   True. First note that $\mathbb{P}(X_0 = a) = 1$. Now:

$$p_{aa}^{(n)} := \sum_{x_1,\ldots,x_{n-1}\in\{a,b\}} p_{ax_1}p_{x_1x_2}\cdots p_{x_{n-1}a} = p_{aa}p_{aa}\cdots p_{aa} = \mu(a)p_{aa}^n = \mathbb{P}(X_0 = X_1 = \cdots = X_n).$$

   We have used here that $p_{ab} = 0$, and thus all the terms of the sum where it appears vanish.

5. For every $x \in \{a, b\}$, $\lim_{n\to\infty} p_{ax}^{(n)} = 0$.

   False. Since $p_{aa}^{(n)} + p_{ab}^{(n)} = 1$, this equality must hold in the limit when $n \to \infty$ as well. It is easy to prove that, in fact, $\lim_{n\to\infty} p_{ab}^{(n)} = 1$.

Solution 1.3

1. $P + Q$ (sum):

   Never a transition matrix. Since each row of P and Q sums to 1, the sum of the elements of a row of $P + Q$ will sum to $1 + 1 = 2$, which does not satisfy the condition that each row should sum to 1.

2. PQ (product):

   Always a transition matrix. Since all elements will be non-negative, we must check that the rows sum up to one. Indeed, fixing a row i and letting the size of the matrices be $n \times n$, we have:

$$\sum_{j=1}^{n}(PQ)_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} p_{ik}q_{kj} = \sum_{k=1}^{n} p_{ik}\left(\sum_{j=1}^{n} q_{kj}\right) = \sum_{k=1}^{n} p_{ik} = 1.$$

   Note: This transition matrix corresponds to the Markov chain consisting of taking one step according to each set of transition probabilities.

3. P^t (transposition):

   Not necessarily a transition matrix. As a counterexample, consider the matrix in Quiz 1.1.

   Note: The transpose will be a transition matrix if and only if the matrix is doubly stochastic, meaning its columns also sum up to 1.

4. P^{-1} (inverse):

   Not necessarily a transition matrix. To begin with, P^{-1} might not exist. Even if it does, one can easily find counterexamples that violate one or both of the defining properties.

5. $\frac{1}{e}\exp(P)$ (rescaled exponential):

   Always a transition matrix. Indeed, $\exp(P)$ is an infinite sum of matrices with non-negative terms, so it has non-negative terms. It remains to show that its columns sum to 1. For a fixed i, we have:

$$\sum_{j=1}^{n}(\exp(P))_{ij} = \sum_{j=1}^{n}\sum_{k=0}^{\infty}\frac{(P^k)_{ij}}{k!} = \sum_{k=0}^{\infty}\sum_{j=1}^{n}\frac{(P^k)_{ij}}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!} = e.$$

   Note: Try to think of the probabilistic interpretation of this transition matrix after studying the chapter on continuous-time Markov chains.

6. $\frac{1}{2}(P^{10} + Q^{11})$:

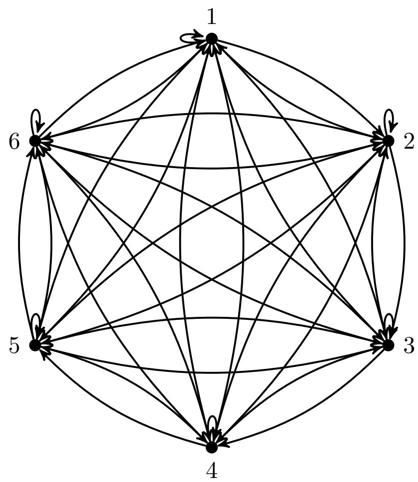
   Always a transition matrix. Both P^{10} and Q^{11} are powers of transition matrices, so they remain transition matrices. Taking their average also results in a matrix whose row sums remain 1, and the elements remain non-negative.

Solution 1.4 [Markov chains]

i) (a) X^1 is a Markov chain, as shown in (b).

(b) The initial distribution of X_0^1 is uniform on S, i.e. $\mu = 1/6 \cdot \left(\delta^1 + \ldots + \delta^6\right)$. It is easy to check that the transition probability $P = (p_{ij})_{i,j \in S}$ is given by $p_{ij} = \frac{1}{6}$, for all i and j.

(c) The corresponding graph is a complete graph where all arrows have a weight of $\frac{1}{6}$:



(d) Since the random variables X_n^1 are independent and uniformly distributed in $\{1, ..., 6\}$, we have that $p_{ij}^{(n)} = \frac{1}{6}$.

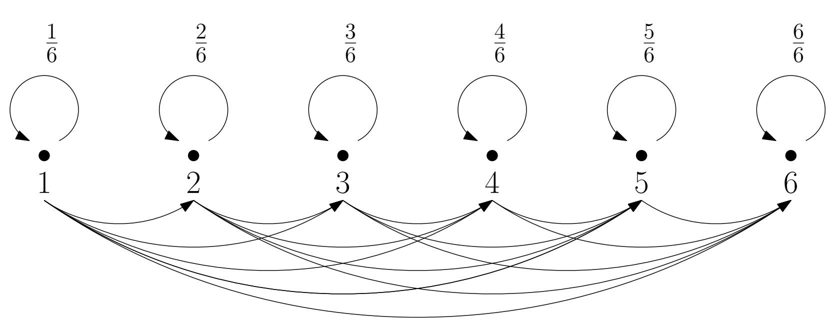
ii) (a) X^2 is a Markov chain, as shown in (b).

(b) We determine the initial distribution μ and a transition probability P such that $X^2 \sim \mathrm{MC}(\mu, P)$. The transition probability $P = (p_{ij})_{i,j \in S}$ is given by

$$p_{ij} = \begin{cases} 0 & \text{if } j < i, \\ \frac{i}{6} & \text{if } j = i, \\ \frac{1}{6} & \text{if } j > i. \end{cases}$$

Indeed, if $X_{n-1}^2 = i$, then we have $X_n = i$ if and only if $\xi_n \leq i$, which happens with probability $i/6$, and we have $X_n = j$ for $j > i$ if and only if $\xi_n = j$, which happens with probability $1/6$. The initial distribution of X_0^2 is, once again, uniform on S, i.e. $\mu = 1/6 \cdot \left(\delta^1 + \ldots + \delta^6\right)$.

(c) We can represent P by the following weighted graph.



Here, all weights on the directed edges (i, j) with $j > i$ are equal to $1/6$.

(d) For every $i, j \in \{1, \dots, 6\}$ and every $n \geq 1$, we have

$$p_{ij}^{(n)} = 0 \quad \text{if } j < i,$$
$$p_{ij}^{(n)} = \left(\frac{i}{6}\right)^n \quad \text{if } j = i,$$
$$p_{ij}^{(n)} = \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n \quad \text{if } j > i.$$

iii) (a) X^3 is a not Markov chain. We note that

$$\{X_2^3 = 1, X_3^3 = 6\} = \{X_2^3 = 1, X_3^3 = 6, X_4^3 = 6\} = \{\xi_1 = 1, \xi_2 = 1, \xi_3 = 6\},$$

and so

$$\mathbb{P}[X_2^3 = 1, X_3^3 = 6] = \mathbb{P}[X_2^3 = 1, X_3^3 = 6, X_4^3 = 6] = (1/6)^3.$$

If $X \sim \mathrm{MC}(\mu, P)$ for some initial distribution μ and transition probability P, then it would follow from Definition 1.3 that

$$p_{66} = \frac{\mathbb{P}[X_2^3 = 1, X_3^3 = 6, X_4^3 = 6]}{\mathbb{P}[X_2^3 = 1, X_3^3 = 6]} = 1, \quad \text{thus } p_{66}^{(n)} = 1, \ \forall n \geq 1.$$

But this contradicts the definition of X since the stochastic process can leave the state 6 with positive probability.

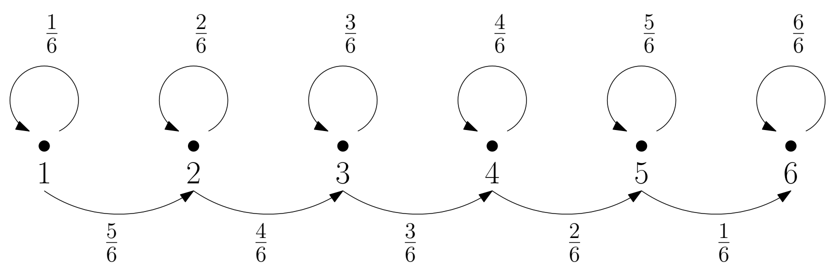
iv) (a) X^4 is a Markov chain, as shown in (b).

(b) We determine the initial distribution μ and a transition probability P such that $X^4 \sim \mathrm{MC}(\mu, P)$. The transition probability $P = (p_{ij})_{i,j \in S}$ is given by

$$p_{ij} = \begin{cases} \frac{6-i}{6} & \text{if } j = i+1, \\ \frac{i}{6} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if $X_{n-1}^4 = i$, then we have $X_n^4 = i+1$ if and only if ξ_n takes a new value, which happens with probability $(6-i)/6$, and we have $X_n^4 = i$ if and only if ξ_n takes no new value, which happens with probability $i/6$. The initial distribution of X_0^4 is $\mu = \delta^1$.

(c) We can represent P by the following weighted graph.



(d) In this case, it requires a bit more work to determine the n-step transition probabilities. We proceed by diagonalizing the matrix P of the transition probability, given by

$$P = \begin{pmatrix} 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/6 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 3/6 & 3/6 & 0 & 0 \\ 0 & 0 & 0 & 4/6 & 2/6 & 0 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix}.$$

Since it is an upper triangular matrix, its eigenvalues are equal to the diagonal entries. By computing the associated eigenvectors, we obtain the matrix Q with the right eigenvectors as columns, given by

$$Q = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad Q^{-1} = \begin{pmatrix} 1 & -5 & 10 & -10 & 5 & -1 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$P^n = Q \cdot \begin{pmatrix} (1/6)^n & 0 & 0 & 0 & 0 & 0 \\ 0 & (2/6)^n & 0 & 0 & 0 & 0 \\ 0 & 0 & (3/6)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & (4/6)^n & 0 & 0 \\ 0 & 0 & 0 & 0 & (5/6)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & (6/6)^n \end{pmatrix} \cdot Q^{-1},$$

which allows to deduce all transition probabilities.

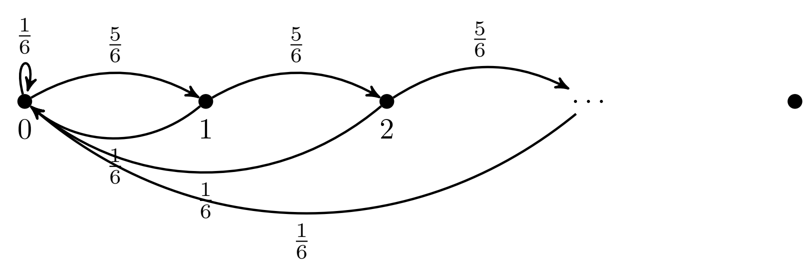
v) (a) X^5 is a Markov chain, as shown in (b).

(b) We determine the initial distribution μ and a transition probability P such that $X^5 \sim \mathrm{MC}(\mu, P)$. The state space is $\mathbb{N}$. The transition probability $P = (p_{ij})_{i,j\in S}$ is given by

$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if } j = 0, \\ \frac{5}{6} & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, we have $X_n^5 = 0$ if and only if $\xi_n = 6$, which happens with probability $1/6$. If $\xi_n \neq 6$, which happens with probability $5/6$, we have $X_n^5 = X_{n-1}^5 + 1$. The initial distribution of X_0^5 is $\mu = 1/6 \cdot \delta^0 + 5/6 \cdot \delta^1$.

(c) We can represent P by the following weighted graph:



(d) For every $n \geq 1$, we have For every $j \in \mathbb{N}$, $i \in \mathbb{N} \cup \{\infty\}$, and every $n \geq 1$, we have

$$p_{ij}^{(n)} = \left(\frac{5}{6}\right)^n \quad \text{if } n = j - i,$$

$$p_{ij}^{(n)} = \frac{1}{6} \cdot \left(\frac{5}{6}\right)^j \quad \text{if } j \leq n - 1,$$

and 0 otherwise.

Solution 1.5 [Deterministic Markov chains]

(a) A deterministic sequence $(x_n)_{n\geq 0}$ is a Markov chain if and only if there exists a function $\Phi : S \to S$ such that for all $n \geq 0$, $x_{n+1} = \Phi(x_n)$.

($\Longleftarrow$): It follows directly that $(x_n)_{n\geq 0}$ is a Markov chain MC(μ, P) with $\mu = \delta^{x_0}$ and transition probability P given by $p_{ij} = \mathbb{1}_{j=\Phi(i)}$.

($\Longrightarrow$): Given $(x_n)_{n\geq 0}$, we define $\Phi : S \to S$ by

$$\Phi(x) = \begin{cases} x_{n+1} & \text{if } \exists n \geq 0 \text{ s.t. } x_n = x, \\ x & \text{if } \forall n \geq 0,\ x_n \neq x. \end{cases}$$

Let $x, y \in S$. The function Φ is well-defined since for every $n \geq 0$ with $x_n = x$,

$$p_{xy} = \frac{\mathbb{P}[x_{n+1} = y, x_n = x]}{\mathbb{P}[x_n = x]} = \mathbb{1}_{x_{n+1}=y},$$

where we used Definition 1.3 in the first inequality and the fact that the sequence is deterministic in the second inequality.

(b) There are three possible choices for the initial distribution, δ^1, δ^2, δ^3. Using the previous exercise, we can choose $\Phi(i) \in \{1, 2, 3\}$ for every $i \in \{1, 2, 3\}$, i.e. there are $3^3 = 27$ possible choices for the transition probability P. Thus, in total there are 81 pairs (μ, P) .

Note that we are counting here the total amount of different pairs (μ, P) that yield deterministic Markov Chains. One can also count the total number of classes of deterministic Markov Chains with respect to equivalence in law of the resulting process, which is 33.