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Applied Stochastic Processes

Solution sheet 10

Solution 10.1

(a)
1. True, by Theorem 6.6 of the lecture notes.
2. False. Theorem 6.6 guarantees uniqueness of solutions to the renewal equation, so $g + 10$ cannot also be a solution.
3. False. Again, by Theorem 6.6, the solution is of the form
$$h + 10 + (h + 10) \star m = g + 10 + 10 \star m = g + 10 + 10F.$$
Thus, $g + 10$ is not a solution.
4. True, as shown in the previous question.
5. False. If it were true, we would have
$$g + F = h + F + g \star F = g + F,$$
implying $m(t) = F(t)$ for all t, which is never the case.

(b)
1. False. For instance, take $F(t) = \mathbb{1}_{\{t\geq 2\}}$ and $h(t) = \mathbb{1}_{\{t\in[0,1]\}}$. Then it is easy to check that the solution g to the renewal equation oscillates between 0 and 1.
2. False. This would be the conclusion of the Smith key renewal theorem if F were non-lattice. However, for lattice distributions, the conclusion fails — see the counterexample in (1).
3. True, by the uniqueness of solutions to the renewal equation and the Smith key renewal theorem.
4. False. See the counterexample from (1).
5. True. This follows directly from applying the renewal equation at $t + s$.

Solution 10.2
We have:

$$\begin{aligned}
m(s+t) = \mathbb{E}[N_{s+t}] &= \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{S_k\in[0,s+t]\}}\right] = \mathbb{E}\left[\sum_{k\geq 1} \left(\mathbb{1}_{\{S_k\in[0,t]\}} + \mathbb{1}_{\{S_k\in(t,s+t]\}}\right)\right] \\
&= \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{S_k\in[0,t]\}}\right] + \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{S_k\in(t,s+t]\}}\right] \\
&= m(t) + \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{S_k\in(t,s+t]\}}\right] \leq m(t) + \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{S_k\in[t,s+t]\}}\right].
\end{aligned}$$

We now aim to show that the second term is bounded above by $1 + m(s)$. The reason it is not exactly equal to $m(s)$ is that we lack control over where the interval $[t, s+t]$ starts relative to the last arrival time. We write:

$$\mathbb{E}\left[\sum_{k\geq 1}\mathbb{1}_{\{S_k\in[t,s+t]\}}\right]=\mathbb{E}\left[\mathbb{1}_{\{S_{N_t+1}\in[t,s+t]\}}+\sum_{k\geq N_t+2}\mathbb{1}_{\{S_k\in[S_{N_t+1},s+t]\}}\right].$$

Here we have separated the first arrival after time t from the remaining arrivals that occur in the interval $[t,t+s]$. We can now bound the first term by 1, and shift the second term to start from zero:

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\{S_{N_t+1}\in[t,s+t]\}}+\sum_{k\geq N_t+2}\mathbb{1}_{\{S_k\in[S_{N_t+1},s+t]\}}\right]&\leq 1+\mathbb{E}\left[\sum_{k\geq 1}\mathbb{1}_{\{S_k\in[0,s+t-S_{N_t+1}]\}}\right]\\&\leq 1+\mathbb{E}\left[\sum_{k\geq 1}\mathbb{1}_{\{S_k\in[0,s]\}}\right]\\&=1+m(s),\end{aligned}$$

where the last inequality follows from the fact that $s+t-S_{N_t+1}\leq s$ almost surely.

Solution 10.3

(a) Let us define $h(t)=\mathbb{1}_{\{t\leq x\}}(1-F(t))$ for $t\geq 0$. Note that $h\geq 0$, it is measurable, continuous a.e. and bounded by 1. Also it vanishes outside the compact interval $[0,x]$. This implies that h is directly Riemann integrable. Since F is non-lattice, by Smith's key renewal theorem it follows that

$$\lim_{t\to\infty}a_x(t)=\frac{1}{\mathbb{E}[T_1]}\int_0^\infty h(t)dt=\frac{1}{\mu}\int_0^x(1-F(t))dt=:G(x).$$

(b) To see that G is a distribution function, we note that

$$\lim_{x\to\infty}G(x)=\frac{1}{\mu}\int_0^\infty\mathbb{P}[T_1>t]dt=\frac{\mathbb{E}[T_1]}{\mu}=1.$$

This means that A_t converges in distribution to a random variable with distribution G.

Solution 10.4

(a) Note that $h\geq 0$ and it is a non increasing function. Also

$$\int_0^\infty h(t)dt=\int_0^\infty\mathbf{P}[U_1>t]dt=\mathbf{E}[U_1]<\infty,$$

which means that h is directly Riemann integrable. Since F is non-lattice and g is solution of the equation $g=h+g*F$, we know by Smith's key renewal theorem that

$$\lim_{t\to\infty}g(t)=\frac{1}{\mathbf{E}[T_1]}\mathbf{E}[U_1]=\frac{\mathbf{E}[U_1]}{\mathbf{E}[U_1]+\mathbf{E}[V_1]}.$$

Solution 10.5

For fixed $k\geq 0$,

$$\begin{aligned}\mathbf{P}[X_n=k]&=\frac{n!}{k!(n-k)!}p_n^k(1-p_n)^{n-k} &(1)\\&=\underbrace{\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n\cdot n\cdot\ldots\cdot n}}_{\xrightarrow{n\to\infty}1}\cdot\frac{1}{k!}\cdot\underbrace{(p_n\cdot n)^k}_{\xrightarrow{n\to\infty}\lambda^k}\big(1-\frac{p_n\cdot n}{n}\big)^{n-k}, &(2)\end{aligned}$$

and since $\frac{p_n \cdot n}{n} \cdot (n-k) \to \lambda$, one has $(1 - \frac{p_n \cdot n}{n})^{n-k} \to e^{-\lambda}$ as $n \to \infty$. Hence,

$$\mathbf{P}[X_n = k] \longrightarrow \frac{e^{-\lambda}}{k!} \lambda^k = \mathbf{P}[X = k], \tag{3}$$

and for $y \in \mathbb{R}$,

$$F_{X_n}(y) = \mathbf{P}[X_n \leq y] = \sum_{k \leq y} \mathbf{P}[X_n = k] \xrightarrow{n \to \infty} \sum_{k \leq y} \mathbf{P}[X = k] = \mathbf{P}[X \leq y] = F_X(y). \tag{4}$$