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D-MATH
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Applied Stochastic Processes

Solution sheet 11

Solution 11.1

(a)

1. True. By assumption, U is a random variable taking values in $[0,5]$. Therefore, the random variable
$$\delta_U : \begin{cases} \Omega & \to \mathcal{M} \\ \omega & \mapsto \delta_{U(\omega)} \end{cases}$$
is well-defined since for all $u \in [0,5]$, δ_u is a σ-finite measure on $([0,5], \mathcal{B}([0,5]))$ taking values in $\{0,1\}$. Hence, δ_U is point process on $([0,5], \mathcal{B}([0,5]))$.
2. True. For all $u \in [0,5]$, $2 \cdot \delta_u$ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ taking values in $\{0,2\}$. As in (b), we deduce that $2 \cdot \delta_U$ is a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
3. True. The proof is analogous as 1. using the random variable $2U$.
4. False. U is not measure-valued.
5. False. The measure takes the value $\frac{1}{2}$ on the set $[0,5]$.

(b)

1. False. $\varphi_X(t) = e^{\lambda(e^{it}-1)}$.
2. True. This is the Poisson Central Limit Theorem.
3. True. One has that:
$$\lim_{n \to \infty} \mathbf{P}(|X_n| \le \varepsilon) \ge \lim_{n \to \infty} \mathbf{P}(X_n = 0) = \lim_{n \to \infty} e^{-\frac{1}{n}} = 1.$$
4. False. Geometric random variables are the only discrete random variable with the loss of memory property.
5. True. We can readily compute both quantities and check they are indeed equal to $\left(\frac{1-e^{-2}}{2}\right)^2$. Alternatively, we can use a coupling argument. Let $K \sim \text{Poisson}(2)$ and let X_n be a family of i.i.d Bernuilli random variables with parameter $\frac{1}{2}$.
Now define:
$$M = \sum_{j=1}^{K} X_j \quad N = \sum_{j=1}^{K} 1 - X_j$$
As seen in class, these two random variables are independent and have a Poisson(1) distribution. We now let $Y_1 = 1 - X_1$ and $Y_j = X_j$ for all $j \ge 2$. Note that the variables Y_n are also i.i.d Bernuilli distributed, but we changed the value of first one. Now define:
$$M' = \sum_{j=1}^{K} Y_j \quad N' = \sum_{j=1}^{K} 1 - Y_j$$
Naturally, $M', N' \sim \text{Poisson}(1)$, but they are coupled with M and N in such a way so that they have different parity pairwise. Now it is easy to check that:
$$\{M \text{ is odd, } N \text{ is odd}\} = \{M' \text{ is even, } N' \text{ is even, } M' + N' \neq 0\}$$

Solution 11.2

(a) Let $n \geq 0$. Using the independence of $X_1, \ldots, X_k$ in the first equality and their Poisson distribution in the second equality, we obtain

$$\begin{aligned}
\mathbb{P}[X_1 + \ldots + X_k = n] &= \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots i_k = n}} \mathbb{P}[X_1 = i_1] \cdots \mathbb{P}[X_k = i_k] \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots+i_k = n}} \frac{{\lambda_1}^{i_1}}{i_1!} \cdots \frac{{\lambda_k}^{i_k}}{i_k!} \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \frac{1}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ \text{s.t. } i_1+\ldots+i_k = n}} \binom{n}{i_1, \ldots, i_k} {\lambda_1}^{i_1} \cdots {\lambda_k}^{i_k} \\
&= e^{-(\lambda_1+\ldots+\lambda_k)} \frac{(\lambda_1 + \ldots + \lambda_k)^n}{n!},
\end{aligned}$$

which shows that $X_1 + \ldots + X_k \sim \mathrm{Pois}(\lambda_1 + \ldots \lambda_k)$.

(b) For $k \geq 1$, define the partial sums $\bar{X}_k := \sum_{i=1}^k X_i$. We first note that $(\bar{X}_k)_{k \geq 1}$ is almost surely a monotone sequence and thus converges almost surely. Hence, $\bar{X}_\infty := \sum_{i=1}^\infty X_i$ is a well-defined random variable taking values in $\mathbb{N} \cup \{+\infty\}$, and we are left with determining its distribution.

Case 1: $\lambda = \sum_{i=1}^\infty \lambda_i = \infty$. In this case, a union bound implies that

$$\mathbb{P}[\bar{X}_\infty < \infty] = \mathbb{P}[\exists I \geq 1, \forall i > I : X_i = 0] \leq \sum_{I \geq 1} \mathbb{P}[\forall i > I : X_i = 0] = \sum_{I \geq 1} \exp(-\underbrace{\sum_{i > I} \lambda_i}_{=\infty}) = 0.$$

Hence, $\bar{X}_\infty = \infty$ almost surely.

Case 2: $\lambda = \sum_{i=1}^\infty \lambda_i < \infty$. From part (a), we know that $\bar{X}_k$ is Poisson-distributed with parameter $\sum_{i=1}^k \lambda_i$. Hence, for all $n \geq 0$,

$$\mathbb{P}[\bar{X}_k = n] = \exp\big(-\sum_{i=1}^k \lambda_i\big) \cdot \frac{(\sum_{i=1}^n \lambda_i)^n}{n!} \longrightarrow \exp(-\lambda) \cdot \frac{\lambda^n}{n!} \quad \text{as } k \to \infty,$$

and so, $\bar{X}_\infty$ is $\mathrm{Pois}(\lambda)$-distributed.

Solution 11.3

(a) We will prove that the preimage of $C_{k,B}$ under $\widetilde{M}$ for any $k \in \mathbb{N}$ and $B \in \mathcal{E}$ is in $\mathcal{F}$. We have:

$$\widetilde{M}^{-1}(C_{k,B}) = \{\omega \in \Omega \mid \widetilde{M}_\omega \in C_{k,B}\} = (G \cap M^{-1}(C_{k,B})) \cup (G^c \cap \{0 = k\}) \in \mathcal{F}$$

where in the last step we used the fact that M is measurable. □

(b) Let $\widetilde{M_1}, \widetilde{M_2}$ be the processes corresponding to G_1, G_2 respectively. Then

$$\mathbf{P}[\widetilde{M_1} \neq \widetilde{M_2}] \leq \mathbf{P}[(G_1 \cap G_2)^c] = 0.$$

This directly implies that, $\forall A \in \mathcal{B}(M)$, we have that $\mathbf{P}(\widetilde{M_1} \in A) = \mathbf{P}(\widetilde{M_2} \in A)$.

Solution 11.4
To show $(i) \iff (ii)$, we note that by definition,

$$P_M = P_{M'} \iff \forall A \in \mathcal{B}(\mathcal{M}), P_M(A) = P_{M'}(A) \iff \forall A \in \mathcal{B}(\mathcal{M}), \mathbf{P}[M \in A] = \mathbf{P}[M' \in A].$$

The implications $(ii) \implies (iii) \implies (iv)$ are clear by inclusion.

To show $(iii) \implies (ii)$, we use Dynkin's lemma. The family

$$\mathcal{B} := \{\{\eta : \eta(B_1) = n_1, \dots, \eta(B_k) = n_k\} : k \geq 1; B_1, \dots, B_k \in \mathcal{E} \text{ disjoint }; n_1, \dots, n_k \in \mathbb{M}\} \subset \mathcal{B}(\mathcal{M})$$

is a π-system and $\sigma(\mathcal{B}) = \mathcal{B}(\mathcal{M})$ by definition. The family

$$\mathcal{D} := \{A \in \mathcal{B}(\mathcal{M}) : \mathbf{P}[M \in A] = \mathbf{P}[M' \in A]\}$$

is a Dynkin-system and it contains $\mathcal{B}$ by assumption. Hence, we conclude using Dynkin's lemma that $\mathcal{D} = \mathcal{B}(\mathcal{M})$ and so (ii) holds.

To show $(iv) \implies (iii)$, we consider $B_1, \dots, B_k$ and $n_1, \dots, n_k$ for some $k \geq 1$ and define the disjoint sets

$$C_1 = B_1,\ C_2 = B_2 \setminus B_1,\ \dots,\ C_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i.$$

(iii) then follows from (iv) by summing over all possible ways how the points could be distributed over the disjoint sets $C_1, \dots, C_k$ under the constraints $M(B_1) = n_1, \dots, M(B_k) = n_k$.