Ritvik Radhakrishnan
Mateo Rodriguez Polo

Applied Stochastic Processes

Solution sheet 12

Solution 12.1

(a)
1. True. For $B \in \mathcal{B}(\mathbb{R})$,
$$T\#\mu(B) = \mu\left(T^{-1}(B)\right) = \mu(B \times [0,2]) = 2 \cdot \mathrm{Leb}_{\mathbb{R}}(B).$$
Hence, $T\#\mu$ is σ-finite and $T\#M$ is a Poisson point process on $\mathbb{R}$ with intensity measure $T\#\mu = 2 \cdot \mathrm{Leb}$.
2. False. For $B \in \mathcal{B}(\mathbb{R})$,
$$T\#\mu(B) = \mu(B \times \mathbb{R}) = \begin{cases} 0 & \text{if } \mathrm{Leb}_{\mathbb{R}}(B) = 0, \\ \infty & \text{if } \mathrm{Leb}_{\mathbb{R}}(B) > 0. \end{cases}$$
Hence, $T\#\mu$ is not σ-finite and $T\#M$ is a not a Poisson point process on $\mathbb{R}$.
3. True. For $B \in \mathcal{B}([0,1])$,
$$T\#\mu(B) = 4 \cdot \mathrm{Leb}_{[0,1]}(B).$$
Hence, $T\#\mu$ is σ-finite and $T\#M$ is a Poisson point process on $[0,1]$ with intensity measure $T\#\mu = 4 \cdot \mathrm{Leb}$.
4. True. We note that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathcal{C}^1$-diffeomorphism and $T^{-1}(y_1, y_2) = (y_2/2, y_1/2)$. Therefore, for all $(y_1, y_2) \in \mathbb{R}^2$, $|\det(dT^{-1}(y_1, y_2))| = 1/4$. Hence, $T\#\mu = 1/4 \cdot \mathrm{Leb}$ is σ-finite and $T\#M$ is a Poisson point process on $\mathbb{R}$ with intensity measure $T\#\mu = 1/4 \cdot \mathrm{Leb}$.
5. True. By the restriction theorem, the restricted processes $M_{[0,1]^2}$, $M_{[0,2]^2}$, and $M_{[2,3]^2}$ are Poisson point processes with intensity measure Leb (on the subsets). Again by the restriction theorem, $M_{[0,1]^2}$ is independent of $M_{[2,3]^2}$ and $M_{[0,2]^2}$ is independent of $M_{[2,3]^2}$ (note that $M(\{(2,2)\}) = 0$ a.s.). The restricted processes $M_{[0,1]^2}$ and $M_{[0,2]^2}$ are *not* independent. For example, it can be seen by noticing that $M_{[0,2]^2}([0,2]^2) = 0$ implies $M_{[0,1]^2}([0,1]^2) = 0$.

(b)
1. True. This is immediate from the definition of product measure on rectangles.
2. True. The product measure of two σ-finite measures taking values in $\mathbb{N} \cup \{\infty\}$ is itself a σ-finite measure taking values in $\mathbb{N} \cup \{\infty\}$.
3. True. Showing that N is a point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is equivalent to showing that for all $C \in \mathcal{E} \otimes \mathcal{F}$, $N(C)$ is a random variable. By applying Dynkin's lemma, it actually suffices to show that $N(C)$ is a random variable for sets of the form $C = A \times B$ with $A \in \mathcal{E}$ and $B \in \mathcal{F}$ (which form a π-system). But in this case, we have by the definition of the product measure $N(A \times B) = M(A) \cdot M'(B)$, which is a product of two random variables and thus a random variable.
4. False. No, N is not a Poisson point process. To illustrate this, we consider the following example: Let $E = F = \mathbb{R}$, $\mu = \nu = \mathrm{Leb}$, and take M and M' to be independent. Then
$$\left\{N([1,2]^2) = 1, N([1,2] \times [3,4]) = 0\right\} = \left\{M([1,2]) = 1, M'([1,2]) = 1, M'([3,4]) = 0\right\},$$
and so
$$\mathbb{P}\left[N([3,4]^2) = 0 | N([1,2]^2) = 1, N([1,2] \times [3,4]) = 0\right] = 1 \neq \mathbb{P}\left[N([3,4]^2) = 0\right],$$
which shows that N cannot be a Poisson point process as it contradicts the independence property on disjoint sets.

5. False. See 4.

Solution 12.2

(a) To see that the restricted measures are diffuse, it suffices to note that for every $i \in \mathbb{N}$,

$$\mu_i(\{x\}) \leq \mu(\{x\}) = 0, \quad \forall x \in E_i,$$

and so μ_i is diffuse.

The fact that $M_{E_1}, M_{E_2}, \ldots$ are independent Poisson point processes with respective intensities $\mu_{E_1}, \mu_{E_2}, \ldots$ follows directly from the restriction property in Section 6.8.

(b) Fix any $i \in \mathbb{N}$. Since $\mu(E_i) < \infty$, we can use Proposition 6.10 to the explicitly construct a Poisson point process with intensity measure μ_{E_i} as

$$\widetilde{M}_{E_i} = \sum_{j=1}^{Z} \delta_{X_j},$$

where $Z \sim \text{Pois}(\mu(E_i))$ and $X_j \sim \frac{\mu_{E_i}(\cdot)}{\mu_{E_i}}$, $j \geq 1$, are independent. Since being simple is a property of the law and $P_{\widetilde{M}_{E_i}} = P_{M_{E_i}}$ by Proposition 6.14, it suffices to prove that $\widetilde{M}_{E_i}$ is almost surely simple. To this end, we compute

$$\mathbb{P}[\widetilde{M}_{E_i} \text{ is not simple}] \leq \mathbb{P}[\exists j \neq k : X_j = X_k] \leq \sum_{j \neq k} \mathbb{P}[X_j = X_k] = 0,$$

where we used in the last equality that

$$\mathbb{P}[X_j = X_k] = \int_{E_i} \underbrace{\mathbb{P}[X_j = x]}_{=0} \frac{\mu_{E_i}(dx)}{\mu(E_i)}$$

by the independence of X_j and X_k. This concludes that $\widetilde{M}_{E_i}$ and thereby M_{E_i} is almost surely simple.

(c) Since $\mathbf{P}[M_{E_i} \text{ is simple}] = 1$ by part (b), we deduce that

$$\mathbf{P}[M \text{ is simple}] = \mathbf{P}[\bigcap_{i=1}^{\infty} \{M_{E_i} \text{ is simple}\}] = 1.$$

Solution 12.3

(a) We consider the map $T : \mathbb{R}^d \to [0, \infty)$ defined by $T(x) = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, which is a continuous function, and so it is measurable. Since $T\#\mu$ is σ-finite (by considering the sequence $([0, n])_{n \geq 1}$), the mapping theorem implies that $T\#M$ is a Poisson point process on $[0, \infty)$ with intensity measure $T\#\mu$.

Let $s \geq r \geq 0$. Then we have

$$T\#\mu([r, s]) = \mu\big(T^{-1}([r, s])\big) = \mu\big(B_s \setminus B_r\big) = \lambda \cdot (|B_s| - |B_r|) = \lambda \cdot \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \cdot (s^d - r^d).$$

More generally, $T\#\mu(B) = \lambda \cdot \text{Leb}(T^{-1}(B))$ for $B \in \mathcal{B}([0, \infty))$.

(b) Fix a sequence $(r_k)_{k\geq 0}$ with $|B_{r_k}| = k$. Using the restriction property from Section 6.8, we note that $\big(M(B_{r_k} \setminus B_{r_{k-1}})\big)_{k\geq 1}$ is a sequence of independent, identically distributed random variables with

$$M(B_{r_1} \setminus B_{r_0}) \sim \text{Pois}(\lambda).$$

Hence, by the strong law of large numbers, we have almost surely

$$\lim_{r\to\infty} \frac{M(B_r)}{|B_r|} = \lim_{n\to\infty} \frac{M(B_{r_n})}{|B_{r_n}|} = \lim_{n\to\infty} \frac{\sum_{k=1}^n M(B_{r_k} \setminus B_{r_{k-1}})}{n} = \mathbb{E}[M(B_{r_1} \setminus B_{r_0})] = \lambda.$$

Solution 12.4

$\Rightarrow$) Since the singletons belong to $\mathcal{B}(E)$, we can define, for all $x \in E$, $N_x = M(\{x\})$, which by the definition of Poisson point process already have the required law. The fact that the family $(N_x)_{x\in E}$ is independent follows directly from the independence of Poisson point processes on disjoint sets.

$\Leftarrow$) Let $B_1, \ldots, B_k \in \mathcal{P}(E)$ be disjoint. For all i,

$$M(B_i) = \sum_{x\in B_i} M(\{x\}) = \sum_{x\in B_i} N_x \tag{1}$$

because $\{x\} \in \mathcal{P}(E)$ for all x. This directly implies that $M(B_i) \sim \text{Poi}\left(\sum_{x\in B_i} \mu(\{x\})\right) = \text{Poi}\left(\mu(B_i)\right)$, since the sum of Poisson random variables is again Poisson.

It remains to prove the independence of the random variables $M(B_1), ..., M(B_k)$. But this follows from expression (1) and the fact that all the N_x are independent.