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Applied Stochastic Processes

Solution sheet 13

Solution 13.1

(a)

1. True. This is exactly the marking theorem from Section 7.10.
2. True. We can define a probability measure $\bar{\nu}$ by $\bar{\nu}(B) = \frac{\nu(B)}{\nu(F)}$ for $B \in \mathcal{F}$ and consider a Poisson point process M on E with intensity measure $\nu(F) \cdot \mu$. Then by the marking theorem, the marked process $\overline{M}$ is a Poisson point process on $E \times F$ with intensity measure $(\nu(F) \cdot \mu) \otimes \bar{\nu} = \mu \otimes \nu$. The equality of the two measures can be obtained by first noticing that they agree on sets of the form $C = A \times B$ for $A \in \mathcal{E}$ and $B \in \mathcal{F}$ and then applying Dynkin's lemma.
3. False. If the measure ν is not σ-finite, then the product measure need not be σ-finite either.
4. False. Take A and μ such that $0 < \mu(A) < \infty$. Let $A \in \mathcal{E}$ where $\mu(A) > 0$. Denote the two marked processes as M_1 and M_2, respectively. Then we have that:

$$\mathbf{P}(M_1(A \times F) = 0 \mid M_2(A \times F) = 0) = 1,$$

while

$$\mathbf{P}(M_1(A \times F) = 0) \neq 1.$$

5. True. The original process is just the projection on the first component of the marked process.

(b)

1. False. The process $M - M'$ can take negative values with positive probability as soon as μ is not identically 0.
2. False. Consider for instance $\mu = \text{Leb}$. The distribution of $M^2([0,1])$ is clearly not Poisson, since it is supported on the squared numbers instead of being supported on $\mathbb{N}$.
3. False. In the same setting as the previous subquestion, we see that the process is not supported on $\mathbb{N}$.
4. True. The distribution of $(M + M')(B)$ is clearly $\text{Poi}(2\mu(B))$, as the sum of independent Poisson random variables is again poisson.

   Given disjoint sets $B_1, ..., B_k$, the independence of $M(B_1) + M'(B_1), ..., M(B_k) + M'(B_k)$ follows directly from the independence of $M(B_1), ..., M(B_k)$ and $M'(B_1), ..., M'(B_k)$.
5. False. It is easy to see that this process is the same as the one featuring in Exercise 12.1 (b) 4, where we concluded that the process is indeed a point process but not a Poisson one.

Solution 13.2

(a) Thanks to the the mapping theorem and the fact that $x \mapsto \frac{1}{x}$ is measurable, it suffices to show that $T_{\#}\mu$ is σ-finite.Consider the following partition of $\mathbb{R}$, where the divergence of the harmonic series guarantees that it actually covers the whole real line:

$$\mathbb{R} = \{0\} \cup \bigcup_{n \in \mathbb{N}_{\geq 1}} A_n \quad \text{where} \quad A_n = \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

- $T_{\#}\mu(\{0\}) = \mu(T^{-1}(\{0\})) = \mu(\emptyset) = 0 < \infty$
- $\forall n \in \mathbb{N} \quad T_{\#}\mu(A_n) = \mu(T^{-1}(A_n)) = \mu((-(n+1), -n] \cup [n, n+1))) = 2 < \infty$

This concludes the proof for σ-finiteness and thus T

(b) Let $A \in \mathcal{B}(\mathbb{R})$. By definition:

$$T_{\#}\mu(A) = \mu(T^{-1}(A))$$

Since T^{-1} is a piecewise C^1-diffeomorphism, we can apply the change of variables formula:

$$\text{Leb}(T^{-1}(A)) = \int_A \left|\det DT^{-1}(x)\right| \, d(\text{Leb})^1(x)$$

In this case:

$$D(T^{-1})(x) = \left(\frac{d}{dx}\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{and} \quad \left|\det D(T^{-1})(x)\right| = \frac{1}{x^2}$$

Thus, the intensity is given by the finite measure defined as

$$T_{\#}\mu(A) = \int_A \frac{1}{x^2} \, dx.$$

Solution 13.3

(a) Let us consider the marked process $\overline{M} = \sum_i \delta_{(X_i, R_i)}$. By the marking theorem, this is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure $\mu \otimes \rho$. In this process, each point (x, r) of the space corresponds to the ball $B(x, r)$. Note that the number of balls that intersect the origin is given by the number of points in the set $A = \{(x, r) \in \mathbb{R}^d \times \mathbb{R}^+ : |x| < r\}$. In other words $N_0 = \overline{M}(A)$. To see that A is measurable, we note that $A := f^{-1}((0, \infty))$ for the measurable function $f((x, r)) = r - |x|$. This implies that N_0 is a well defined random variable and that $N_0 \sim \text{Poisson}((\mu \otimes \rho)(A))$. We know by Fubini's Theorem that

$$(\mu \otimes \rho)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_A(y)(\mu \otimes \rho)(dy) = \int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx),$$

which shows what we wanted.

(b) First, note that $\{\mathcal{O} = \mathbb{R}^d\} = \bigcap_{N \geq 1} \{\mathcal{O} \supset \overline{B(0, N)}\}$. By compactness of $\overline{B(0, N)}$,

$$\{\mathcal{O} \supset \overline{B(0, N)}\} = \bigcup_{I \geq 1} \left\{ \overline{B(0, N)} \subset \bigcup_{1 \leq i \leq I} B(X_i, R_i) \right\},$$

and for every $I \in \mathbb{N}$,

$$\left\{\overline{B(0,N)} \subset \bigcup_{1\le i\le I} B(X_i,R_i)\right\} = \bigcup_{n\ge 1}\left\{\overline{B(0,N)} \subset \bigcup_{1\le i\le I} B(X_i,R_i) \text{ and } R_i \ge \frac{1}{n}, \forall i \le I\right\}.$$

Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, we have

$$\left\{\overline{B(0,N)} \subset \bigcup_{1\le i\le I} B(X_i,R_i) \text{ and } R_i \ge \frac{1}{n}, \forall i \le I\right\} = \bigcap_{q\in\overline{B(0,N)}\cap\mathbb{Q}^d}\ \bigcup_{1\le i\le I} \underbrace{\{q \in B(X_i,R_i)\}}_{=\{|X_i-q|<R_i\}}.$$

In summary, we have shown that $\{\mathcal{O} = \mathbb{R}^d\}$ can be written in terms of countable unions and intersections of measurable sets, and is therefore measurable. Second, we know by Fubini's theorem that

$$\int_{\mathbb{R}^d}\int_{|x|}^{\infty} \rho(dr)\mu(dx) = \int_0^{\infty}\int_{B(0,r)} \mu(dx)\rho(dr) = \pi_d \int_0^{\infty} r^d \rho(dr)$$

Hence,

$$\mathbf{P}[0 \notin \mathcal{O}] = \mathbf{P}[N_0 = 0] = \exp\left(-\pi_d \int_0^{\infty} r^d \rho(dr)\right).$$

Suppose that $\mathbf{P}[\mathcal{O} = \mathbb{R}^d] = 1$. Then $P[0 \in \mathcal{O}] = 1$, and we deduce from the last expression that $\int_0^{\infty} r^d\rho(dr) = \infty$. To prove the converse, assume that $\int_0^{\infty} r^d\rho(dr) = \infty$. As a preliminary result we first show for any $n \in \mathbb{N}$ that

$$(\mu\otimes\rho)\left(\{(x,r)\in\mathbb{R}^d\times\mathbb{R}^+ : B(0,n)\subset B(x,r)\}\right) = \infty. \tag{1}$$

Since $B(0,n) \subset B(x,r)$ if and only if $r \ge |x| + n$, the left-hand side of equation (1) equals

$$\int_0^{\infty}\int_{\mathbb{R}^d} 1_{\{r\ge|x|+n\}}\mu(dx)\rho(dr) = \pi_d\int_n^{\infty}(r-n)^d\rho(dr).$$

This is bounded below by

$$\pi_d\int_{2n}^{\infty}\left(\frac{r}{2}\right)^d\rho(dr) = \pi_d 2^{-d}\int_0^{\infty} 1_{\{r\ge 2n\}} r^d\rho(dr),$$

proving (1). Since $\overline{M}$ is a Poisson point process with intensity $\mu\otimes\rho$, the ball $B(0,n)$ is almost surely covered even by one of the balls $B(X_i,R_i)$. Since n is arbitrary, it follows that $\mathbf{P}[\mathcal{O} = \mathbb{R}^d] = 1$.

Solution 13.4

(a) Recall by definition that $\mu(B) = \mathbf{E}[M(B)]$. We have

$$\mathcal{L}_M(t1_B) = \mathbf{E}\left[\exp\left(-t\int_E 1_B M(dx)\right)\right] = \mathbf{E}[\exp(-tM(B))].$$

Since $M(B) \geq 0$, the exponential above is bounded by 1. Besides, $M(B) \in L^1(\mathbf{P})$, so we can exchange the derivative and the expectation in the Laplace functional, therefore

$$-\frac{d}{dt}\mathcal{L}_M(t1_B) = \mathbf{E}[M(B)\exp(-tM(B))].$$

It suffices now to take $t = 0$ to conclude.

(b) For all $t > 0$, we have $L_M(t1_B) = \mathbf{E}[\exp(-tM(B))] = \mathbf{E}[1_{\{M(B)=0\}} + e^{-tM(B)}1_{\{M(B)\geq 1\}}]$. By dominated convergence, we get

$$\lim_{t\to\infty}\mathcal{L}_M(t1_B) = \mathbf{E}[1_{\{M(B)=0\}}] + 0 = \mathbf{P}[M(B) = 0].$$

Solution 13.5

If $u(x) = 1_B(x)$ for some $B \in \mathcal{E}$, then $\int u(x)N(dx) = N(B)$. We know that $N(B) : \Omega \to \mathbb{N} \cup \{\infty\}$ is the mapping $\omega \mapsto N(\omega)(B)$, and it is a measurable map by definition. We also know that $\mathbf{E}[N(B)] = \mu(B)$. Then both assertions hold for this choice of u. By linearity we can extend this result to simple functions. Since the limit of measurable functions is measurable and using monotone convergence theorem, we can also extend both assertions to arbitrary $u : E \to \mathbb{R}_0^+$. Now let us consider $u : E \to \mathbb{R}$. We can write $u = u_+ - u_-$ with $u_+, u_- : E \to \mathbb{R}_0^+$, and this implies that $\int u(x)N(dx)$ is a random variable. Assume that $\int |u(x)|\mu(dx) < \infty$. Then we have that

$$\begin{aligned}\mathbf{E}\left[\int u(x)N(dx)\right] &= \mathbf{E}\left[\int u_+(x)N(dx)\right] - \mathbf{E}\left[\int u_-(x)N(dx)\right]\\ &= \int u_+(x)\mu(dx) - \int u_-(x)\mu(dx) = \int u(x)\mu(dx),\end{aligned}$$

which concludes the proof.