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D-MATH
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Applied Stochastic Processes

Solution sheet 14

Solution 14.1

(a)
1. True. This is an application of the three equivalences theorem for the Standard Poisson Process seen in the lectures.
2. False. We can take as a counterexample $(2N_t)_{t\geq 0}$, where $(N_t)_{t\geq 0}$ is a Poisson Process.
3. True. These conditions readily imply those of question 1.
4. False. We can take as a counterexample a Simple Random Walk.
5. True. This is just the definition of a Standard Poisson Process as seen in the lectures.

(b)
1. False. In order to be a Poisson process, the stochastic process $(N_t)_{t\geq 0}$ would also need to be almost surely non-decreasing and right-continuous (see the definition of counting process in Section 7.1).
2. False. Let U be a uniform random variable taking values in $(0, 1]$. Define $(N_t)_{t\geq 0}$ by

$$N_t := \sum_{i=0}^{\infty} \mathbb{1}_{i+U\leq t}.$$

   It follows directly from the definition that $(N_t)_{t\geq 0}$ is a counting process and it makes jumps of size 1 at the times $U, 1+U, 2+U, \ldots$. The process has stationary increments since for any $t > s \geq 0$, the increment $N_t - N_s$ only depends on $t - s$. To show that the increments are *not* independent, it suffices to note that

$$N_{1/2} - N_0 = \mathbb{1}_{0<U\leq 1/2} \quad \text{and} \quad N_1 - N_{1/2} = \mathbb{1}_{1/2<U\leq 1} = 1 - (N_{1/2} - N_0).$$

3. True. By definition, a counting process is almost surely non-decreasing and right-continuous. In particular, left limits almost surely exist due to the monotonicity of the process.
4. True. By definition, a counting process almost surely is non-decreasing and takes values in $\mathbb{N}$. Hence, for any $t \geq 0$, the number of jumps in $[0, t]$ is at most N_t. Since the random variable N_t is almost surely finite, the same holds for the number of jumps.
5. True. The sum of n exponentially distributed random variables with parameter λ is a Gamma(n, λ) distribution.

Solution 14.2

(a) For the choice $\rho(u) = \lambda$ for all $u \geq 0$, we obtain for $0 \leq s < t$,

$$\int_s^t \rho(u)du = \lambda(t - s),$$

and so $N_t - N_s \sim \text{Poisson}(\lambda(t - s))$. Hence, it follows from part (iii) of Theorem 7.2 or part (ii) of Theorem 7.3 from the 2023 lecture notes that $(N_t)_{t\geq 0}$ is a Poisson process with rate λ.

(b) In general, the increments are *not* stationary. In part (a), we have seen that the increments are stationary if ρ is constant. Conversely, if ρ is not constant, we can choose $u, v \geq 0$ such that $\rho(u) > \rho(v)$. Then for some $h > 0$ sufficiently small,

$$\int_u^{u+h} \rho(u)du > \int_v^{v+h} \rho(v)dv,$$

and so the increments $N_{u+h} - N_u$ and $N_{v+h} - N_v$ do not have the same distribution.

(c) The intensity measure μ_ρ of the Poisson point process M is defined by

$$\mu_\rho(B) = \int_B \rho(u)du$$

for $B \in \mathcal{B}(\mathbb{R}_+)$.

(d) In general, S_1 and $S_2 - S_1$ are *not* independent as the following result shows.

Claim: S_1 and $S_2 - S_1$ are independent if and only if ρ is constant.

($\Longleftarrow$): If ρ is constant, then by part (a), $(N_t)_{t\geq 0}$ is actually a Poisson process with rate $\lambda = \rho$. Hence, the inter-arrival times are independent $\mathrm{Exp}(\lambda)$-distributed random variables.

($\Longrightarrow$): Let $s, t \geq 0$. For every $\epsilon \in (0, s)$, we have by the independence of the increments that

$$\mathbb{P}[t < S_1 \leq t + \epsilon] = \mathbb{P}[N_t = 0, N_{t+\epsilon} - N_t \geq 1] = \mathbb{P}[N_t = 0] \cdot \mathbb{P}[N_{t+\epsilon} - N_t \geq 1],$$

and

$$\begin{aligned}\mathbb{P}[S_2 - S_1 > s, t < S_1 \leq t + \epsilon] &= \mathbb{P}[N_t = 0, N_{t+\epsilon} - N_t = 1, N_{t+s} - N_{t+\epsilon} = 0] \\ &= \mathbb{P}[N_t = 0] \cdot \mathbb{P}[N_{t+\epsilon} - N_t = 1] \cdot \mathbb{P}[N_{t+s} - N_{t+\epsilon} = 0].\end{aligned}$$

By assumption, S_1 and $S_2 - S_1$ are independent and so we have

$$\mathbb{P}[S_2 - S_1 > s] = \mathbb{P}[S_2 - S_1 > s \,|\, t < S_1 \leq t + \epsilon] = \mathbb{P}[N_{t+s} - N_{t+\epsilon} = 0] \cdot \frac{\mathbb{P}[N_{t+\epsilon} - N_t = 1]}{\mathbb{P}[N_{t+\epsilon} - N_t \geq 1]}.$$

Letting $\epsilon \to 0$, it follows that

$$\mathbb{P}[S_2 - S_1 > s] = \exp\left(-\int_t^{t+s} \rho(u)du\right),$$

where we have used that $\frac{x\cdot e^{-x}}{1-e^{-x}} \to 1$ as $x \to 0$. This is only possible if for all $s \geq 0$, $\int_t^{t+s} \rho(u)du$ does not depend on t. As in part (b), we conclude that ρ must be constant.

Solution 14.3

(a) The function $\rho : [0, +\infty) \to (0, +\infty)$ is continuous, hence integrable, and so R is well-defined and continuous as a function of t. Since ρ is strictly positive, R is strictly increasing as a function of t, hence injective. Finally, since $\int_0^\infty \rho(u)du = +\infty$, R is surjective.

(b) Since R is a continuous, increasing bijection by part (a), $R^{-1} : [0, +\infty) \to [0, +\infty)$ is a well-defined continuous, increasing bijection. In particular, $R^{-1}(0) = 0$. This implies that $(\widetilde{N}_t)_{t\geq 0}$ is a counting process. Furthermore, for any $k \geq 1$ and $0 = t_0 < t_1 < \ldots t_k$, it holds that $0 = R^{-1}(t_0) < R^{-1}(t_1) < \ldots < R^{-1}(t_k)$, and so the independence of the increments of $\widetilde{N}$ follows from the independence of the increments of N. Finally, for $0 \leq s < t$,

$$\widetilde{N}_t - \widetilde{N}_s = N_{R^{-1}(t)} - N_{R^{-1}(s)} \sim \mathrm{Pois}\Big(\underbrace{\int_{R^{-1}(s)}^{R^{-1}(t)} \rho(u)du}_{=t-s}\Big),$$

and so we conclude that $\widetilde{N}$ is a Poisson process with rate 1.

(c) As in part (c), we first note that $(N_t)_{t\geq 0}$ is a counting process with independent increments. Furthermore, for $0 \leq s < t$,

$$N_t - N_s = \widetilde{N}_{R(t)} - \widetilde{N}_{R(s)} \sim \mathrm{Pois}(\underbrace{R(t) - R(s)}_{\int_s^t \rho(u)du}),$$

and so we conclude that N is an inhomogeneous Poisson process with rate ρ.

Remark: Alternatively, it is possible to prove (b) and (c) using the mapping theorem for Poisson point processes from Section 7.9 and the correspondence between Poisson processes and Poisson point processes established in Theorem 7.2 from the 2023 lecture notes (as well as an analogous result for inhomogenous Poisson processes).

Solution 14.4

(a) No. In order to be a Poisson process, the stochastic process $(N_t)_{t\geq 0}$ would also need to be almost surely non-decreasing and right-continuous (see the definition of counting process in Section 7.1).

(b) Let U be a uniform random variable taking values in $(0, 1]$. Define $(N_t)_{t\geq 0}$ by

$$N_t := \sum_{i=0}^{\infty} \mathbb{1}_{i+U\leq t}.$$

It follows directly from the definition that $(N_t)_{t\geq 0}$ is a counting process and it makes jumps of size 1 at the times $U, 1 + U, 2 + U, \ldots$. The process has stationary increments since for any $t > s \geq 0$, the increment $N_t - N_s$ only depends on $t - s$. To show that the increments are *not* independent, it suffices to note that

$$N_{1/2} - N_0 = \mathbb{1}_{0<U\leq 1/2} \quad \text{and} \quad N_1 - N_{1/2} = \mathbb{1}_{1/2<U\leq 1} = 1 - (N_{1/2} - N_0).$$

(c) Yes. By definition, a counting process is almost surely non-decreasing and right-continuous. In particular, left limits almost surely exist due to the monotonicity of the process.

(d) Yes. By definition, a counting process almost surely is non-decreasing and takes values in $\mathbb{N}$. Hence, for any $t \geq 0$, the number of jumps in $[0, t]$ is at most N_t. Since the random variable N_t is almost surely finite, the same holds for the number of jumps.

Solution 14.5

(a) First we will show that almost surely there exists n_0 such that for all $n \geq n_0$ we have

$$T_n \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Set $E_n := \{T_n > \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\}$, then

$$\mathbb{P}[E_n] = \exp\left(-\lambda \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\right) = \left(\frac{\lambda}{n}\right)^{1+\varepsilon},$$

hence $\sum_n \mathbb{P}[E_n] < \infty$ and therefore by Borel-Cantelli, we obtain $\mathbb{P}[\limsup_{n\to\infty} E_n] = 0$. This means that for almost every ω, there is $n_0(\omega)$ such that for all $n \geq n_0(\omega)$ we have

$$\max_{n_0(\omega)\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \max_{n_0(\omega)\leq k\leq n} \log(k/\lambda) = \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Furthermore, we can choose $n_1(\omega) \geq n_0(\omega)$ such that

$$\max_{1 \leq k \leq n_0(\omega)} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n_1(\omega)/\lambda),$$

because log is a monotone function increasing to infinity. Therefore almost surely, there is n_1 such that for all $n \geq n_1$, we have

$$\max_{1 \leq k \leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

(b) We have $\limsup_{t\to\infty} \frac{N_t+1}{t} = \limsup_{t\to\infty} \frac{N_t}{t}$ and

$$\limsup_{t\to\infty} \frac{N_t}{t} \leq \limsup_{t\to\infty} \frac{N_t}{S_{N_t}} = \limsup_{k\to\infty} \frac{k}{S_k} = \lambda,$$

where we used in the last step that by the strong law of large numbers we have $S_k/k \to \frac{1}{\lambda}$ almost surely as $k \to \infty$. This implies that almost surely there is t_0 such that for all $t > t_0$ we have

$$\frac{N_t + 1}{t} \leq (1+\varepsilon)\lambda.$$

(c) Almost surely for t large enough we have

$$L_t \leq \max_{1 \leq k \leq N_t+1} T_k \leq \frac{(1+\varepsilon)}{\lambda} \log\left(\frac{N_t+1}{\lambda}\right) \leq \frac{(1+\varepsilon)}{\lambda} \log(t(1+\varepsilon)),$$

which yields $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \frac{(1+\varepsilon)}{\lambda}$. As $\varepsilon > 0$ was arbitrarily chosen this yields the claim.