ETH Zürich, FS 2025
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Mateo Rodriguez Polo

Applied Stochastic Processes

Solution sheet 2

Solution 2.1

(a)
1. $\mathbb{P}_x(X_1 = y, X_0 = x) = \delta^x(x) \cdot p_{xy} = p_{xy}$.
2. $\mathbb{P}_x(X_1 = y) = \mathbb{P}_x(X_1 = y, X_0 = x) = \delta^x(x) \cdot p_{xy} = p_{xy}$.
3. $P_x(X_1 = x, X_0 = y) = p_{xx}p_{xy}$, which is not equal to p_{xy} in general.
4. $P_x(X_1 = x, X_0 = y) = \delta^x(y) \cdot p_{yx}$, which is not equal to p_{xy} in general.
5. Under the measure $\mathbb{P}_x$, we have $X_0 \sim \delta^x$. Thus, $\mathbb{P}_\mu(X_1 = y) = \mathbb{P}_x(X_1 = y)$.

Thus, the correct answers are 1, 2 and 5.

(b) By definition of the n-step transition probability, we obtain

$$\begin{aligned}\mathbb{P}_x(X_{n+2} = z, X_{n+1} = y, X_n = x) &= p_{xx}^{(n)} \cdot p_{xy} \cdot p_{yz} \\ &= \sum_{x_1,\dots,x_{n-1} \in S} p_{xx_1} \cdot \ldots p_{x_{n-1}x} \cdot p_{xy} \cdot p_{yz}.\end{aligned}$$

Thus, 3. and 5. are equal.

2. is also equal due to the 1-step and homogeneity property of Markov Chains:

$$\begin{aligned}&\mathbb{P}_x(X_{2n+2} = z, X_{2n+1} = y, X_{2n} = x | X_n = x) = \\ &= \mathbb{P}_x(X_{2n+2} = z | X_{2n+1} = y)\mathbb{P}_x(X_{2n+1} = y | X_{2n} = x)\mathbb{P}_x(X_{2n} = x | X_n = x) \\ &= \mathbb{P}_x(X_{n+2} = z | X_{n+1} = y)\mathbb{P}_x(X_{n+1} = y | X_2 = x)\mathbb{P}_x(X_n = x) \\ &= \mathbb{P}_x(X_{n+2} = z, X_{n+1} = y, X_n = x).\end{aligned}$$

On the other hand, we can apply the simple Markov property with $k = n$, $Z = 1$ and $f((X_{k+m})_{m \geq 0}) = \mathbb{1}_{X_{k+2}=z, X_{k+1}=y}$ to obtain

$$\mathbb{P}_x(X_{n+2} = z, X_{n+1} = y | X_n = x) = \mathbf{E}_x(\mathbb{1}_{X_2=z, X_1=y}) = \mathbb{P}_x(X_2 = z, X_1 = y) = p_{xy} \cdot p_{yz}.$$

Thus, 1. and 4. are equivalent to each other but not to $\mathbb{P}_x(X_{n+2} = z, X_{n+1} = y, X_n = x)$.

(c) We have:

1. $\mathbb{P}_1(X_2 = 3) = p_{12} \cdot p_{23} = 1/4$.
2. $\mathbb{P}_1(X_3 = 3) = 0$, since the SRW is at even values at odd times.
3. $\mathbb{P}_1(X_4 = 3) = \binom{4}{1} \cdot 1/16 = 1/4$. We used that every nearest-neighbor path on $\mathbb{Z}$ from 1 to 3 of length 4 does exactly 3 steps "+1" and 1 step "-1". Each such path has probability 1/16 and there are $\binom{4}{1}$ ways to choose the position of the step "-1".
4. $\mathbb{P}_1(X_5 X_6 < 0) = 0$. Two consecutive values of the SRW cannot have opposite signs.
5. $\mathbb{P}_1(X_{2n+1} = 0) = \binom{2n+1}{n} 2^{-2n-1}$ for $n \in \mathbb{N}$. Indeed, we can get to 0 after $2n+1$ steps by summing $n+1$ times "-1" and n times "+1". Each particular way of achieving this has probability 2^-n of happening, and in total there are $\binom{2n+1}{n}$ ways of choosing when the "+1" happen.

Thus, the probabilities that are 0 are 2 and 5.

Solution 2.2 Let us identify the set a, b, c with $1, 2, 3$. Then, from the diagram we can get the following transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

We know that $\mathbb{P}(X_n = a) = p^{(n)}_{a,a} = P^n(1, 1)$. Then we need to calculate P^n. We see that this matrix is diagonalizable since it has different eigenvalues. Indeed, it characteristic equation is given by

$$0 = \det(\lambda I - P) = \lambda \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}(\lambda - 1)(4\lambda^2 + 1)$$

and its eigenvalues are $1, i/2, -i/2$. Hence, there exists an invertible matrix U such that

$$P = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} U^{-1}$$

and then

$$P^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & (i/2)^n & 0 \\ 0 & 0 & (-i/2)^n \end{pmatrix} U^{-1}$$

This implies that $P^n(1, 1) = x + y(i/2)^n + z(-i/2)^n$ for some constants x, y, z. We can calculate the value of these constants by using the first steps of our chain

$$\begin{aligned} 1 &= P^0(1, 1) = x + y + z \\ 0 &= P^1(1, 1) = x + iy/2 - iz/2 \\ 0 &= P^2(1, 1) = x - y/4 - z/4. \end{aligned}$$

This give us $x = 1/5$, $y = (i - 2)/5$ and $z = (2 - i)/5$. Therefore

$$\begin{aligned} P^n(1, 1) &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{i}{2}\right)^n + \frac{2-i}{5}\left(\frac{-i}{2}\right)^n \\ &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) + \frac{2-i}{5}\left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right) \\ &= \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right). \end{aligned}$$

Solution 2.3 Define μ to be the law of X_0 and set

$$p_{xy} = \begin{cases} \mathbb{P}(X_{n+1} = y | X_n = x) & \text{if } \exists n : \mathbb{P}(X_n = x) > 0, \\ \mathbb{1}_{x=y} & \text{otherwise.} \end{cases}$$

By homogeneity, p_{xy} is well-defined. Furthermore, for every $x_0, \ldots, x_n \in S$, we have

$$\begin{aligned} \mathbb{P}(X_0 = x_0, \ldots, X_n = x_n) &= \underbrace{\mathbb{P}(X_0 = x_0)}_{=\mu(x_0)} \cdot \prod_{i=1}^{n} \underbrace{\mathbb{P}(X_i = x_i | X_0 = x_0, \ldots, X_{i-1} = x_{i-1})}_{=\mathbb{P}(X_i = x_i | X_{i-1} = x_{i-1}) = p_{x_{i-1}x_i}} \\ &= \mu(x_0) \cdot p_{x_0 x_1} \cdot \ldots \cdot p_{x_{n-1}x_n}, \end{aligned}$$

where we used the 1-step Markov property and the definitions of μ and P.

It remains to check that P is a transition probability. Let $x \in S$. If there exists $n \geq 0$ such that $\mathbb{P}(X_n = x) > 0$, then

$$\sum_{y \in E} p_{xy} = \sum_{y \in E} \mathbb{P}(X_{n+1} = y | X_n = x) = 1.$$

Otherwise,

$$\sum_{y \in E} p_{xy} = \sum_{y \in E} \mathbb{1}_{x=y} = 1.$$

Solution 2.4

Under $\mathbb{P}_0$, $(X_n)_{n\geq 0}$ is a simple random walk (SRW) starting at 0. For $i \in \mathbb{Z}$ and $k \geq 0$,

$$\begin{aligned}
\mathbb{P}_0\left(Z' = k | X_{10} = i\right) &= \mathbb{P}_0\left[\left(\sum_{n=10}^{20} \mathbb{1}_{X_n = i}\right) = k | X_{10} = i\right] = \mathbb{P}_i\left[\left(\sum_{n=0}^{10} \mathbb{1}_{X_n = i}\right) = k\right] \\
&= \mathbb{P}_0\left[\left(\sum_{n=0}^{10} \mathbb{1}_{X_n = 0}\right) = k\right] = \mathbb{P}_0\left(Z = k\right).
\end{aligned}$$

Above, the second equality follows from the simple Markov property (Theorem 1.5 in the lecture notes with $f(x_0, x_1, \ldots) = \mathbb{1}\{\sum_{j=0}^{10} \mathbb{1}_{x_j = i} = k\}$) and the third equality follows since $(i + X_n)_{n\geq 0}$ is a SRW starting at i (under $\mathbb{P}_0$). Since the right-hand side does not depend on i, it directly follows that

$$\mathbb{P}_0(Z' = k) = \sum_{i \in \mathbb{Z}} \mathbb{P}_0\left(Z' = k | X_{10} = i\right) \cdot \mathbb{P}_0\left(X_{10} = i\right) = \mathbb{P}_0\left(Z = k\right),$$

and so Z and Z' have the same distribution. Furthermore, we see that Z' and X_{10} are independent.The simple Markov property implies that Z and Z' are conditionally independent given $\{X_{10} = i\}$. Therefore, Z and Z' are independent as the following computation shows:

$$\begin{aligned}
\mathbb{P}_0(Z = k, Z' = \ell) &= \sum_{i \in \mathbb{Z}} \mathbb{P}_0(Z = k, Z' = \ell | X_{10} = i) \cdot \mathbb{P}_0(X_{10} = i) \\
&= \sum_{i \in \mathbb{Z}} \mathbb{P}_0(Z = k | X_{10} = i) \cdot \mathbb{P}_0(Z' = \ell | X_{10} = i) \cdot \mathbb{P}_0(X_{10} = i) \\
&= \mathbb{P}_0(Z' = \ell) \cdot \left(\sum_{i \in \mathbb{Z}} \mathbb{P}_0(Z = k | X_{10} = i) \cdot \mathbb{P}_0(X_{10} = i)\right) \\
&= \mathbb{P}_0(Z' = \ell) \cdot \mathbb{P}_0(Z = k).
\end{aligned}$$

Solution 2.5

(a) We establish the inequality by induction on k. For $k = 0$, the inequality is trivial. For $k \geq 1$, it follows from the simple Markov property that

$$\begin{aligned}
&\mathbb{P}_0(H_{-N,N} > k \cdot N) \\
&= \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbb{P}_0(H_{-N,N} > k \cdot N, X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N}) \\
&= \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbb{P}_{x_{(k-1)N}}(H_{-N,N} > N) \cdot \mathbb{P}_0(X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N})
\end{aligned}$$

Since the distance from any $x \in \{-N+1, \dots, N-1\}$ to either N or $-N$ is at most N, it follows that $\mathbb{P}_x(H_{-N,N} \leq N) \geq 2^{-N}$. Thus,

$$\begin{aligned}
&\mathbb{P}_0(H_{-N,N} > k \cdot N) \\
&\leq (1 - 2^{-N}) \cdot \sum_{-N+1 \leq x_1,\dots,x_{(k-1)N} \leq N-1} \mathbb{P}_0(X_1 = x_1, \dots, X_{(k-1)N} = x_{(k-1)N}) \\
&= (1 - 2^{-N}) \cdot \mathbb{P}_0(H_{-N,N} > (k-1) \cdot N) \leq (1 - 2^{-N})^k,
\end{aligned}$$

where we used the induction hypothesis in the last step.

We compute

$$\mathbf{E}_0(H_{-N,N}) = \sum_{\ell=0}^{\infty} \mathbb{P}_0(H_{-N,N} > \ell) \leq \sum_{k=0}^{\infty} N \cdot \underbrace{\mathbb{P}_0(H_{-N,N} > k \cdot N)}_{(1-2^{-N})^k} = N \cdot 2^N.$$

(b) Assume towards a contradiction that $\mathbf{E}_x(H_{-N,N}) = \infty$ for some $x \in \{-N, \dots, N\}$. Without loss of generality, let us assume that x is a non-negative integer. Then $p_{0x}^{(x)} = 2^{-x}$, and so by the simple Markov property

$$\begin{aligned}
\mathbf{E}_0(H_{-N,N}) &\geq \mathbf{E}_0(H_{-N,N} \cdot \mathbb{1}_{X_1=1,\dots,X_x=x}) \\
&= \underbrace{\mathbf{E}_x((H_{-N,N} + x))}_{=\infty} \cdot \underbrace{\mathbb{P}_0(X_1 = 1, \dots, X_x = x)}_{=2^{-x}} = \infty,
\end{aligned}$$

which contradicts the result of (a).

(c) First, we note that by (b), the function $f : \{-N, \dots, N\} \to \mathbb{R}_+$, given by

$$f(x) = \mathbf{E}_x(H_{-N,N}),$$

is well-defined. Moreover, f is even (i.e. $f(x) = f(-x)$) due to the symmetry of the SRW, and it has boundary values $f(-N) = f(N) = 0$. For $x \in \{-N+1, \dots, N-1\}$,

$$\begin{aligned}
f(x) = \mathbf{E}_x(H_{-N,N}) &= \mathbf{E}_x(H_{-N,N} \cdot \mathbb{1}_{X_1=x-1}) + \mathbf{E}_x(H_{-N,N} \cdot \mathbb{1}_{X_1=x+1}) \\
&= \mathbf{E}_{x-1}(H_{-N,N} + 1) \cdot \mathbb{P}_x(X_1 = x-1) + \mathbf{E}_{x+1}(H_{-N,N} + 1) \cdot \mathbb{P}_x(X_1 = x+1) \\
&= (f(x-1) + 1) \cdot \frac{1}{2} + (f(x+1) + 1) \cdot \frac{1}{2} = \frac{f(x-1) + f(x+1)}{2} + 1
\end{aligned}$$

Equivalently, for every $x \in \{-N+1, \dots, N-1\}$,

$$f(x) - f(x-1) = f(x+1) - f(x) + 2.$$

Let $n \geq 0$. Summing over all $x \in \{-n, \dots, n\}$, it follows that

$$\begin{aligned} f(n) - f(-n-1) &= \sum_{x=-n}^{n} (f(x) - f(x-1)) \\ &= \sum_{x=-n}^{n} (f(x+1) - f(x) + 2) = f(n+1) - f(-n) + 2(2n+1). \end{aligned}$$

Thus, since f is even, we obtain

$$f(n) = f(n+1) + (2n+1).$$

Using $f(N) = 0$, we inductively obtain

$$\begin{aligned} f(n) &= \sum_{m=n}^{N-1} (2m+1) = 2 \cdot \left(\sum_{m=n}^{N-1} m \right) + (N-n) \\ &= 2 \cdot \left(\frac{N(N-1)}{2} - \frac{n(n-1}{2} \right) + (N-n) \\ &= N^2 - n^2. \end{aligned}$$

In particular, $f(0) = N^2$, which is what we wanted to show.

Remark: Another strategy would be to show that the function $g : \{-N, \dots, N\} \to \mathbb{R}_+$, defined by

$$g(x) = f(x) + x^2,$$

is harmonic in the interior of $\{-N, \dots, N\}$ and satisfies $g(-N) = g(N) = N^2$. Using the uniqueness of the solution to the Dirichlet problem, i.e. the fact that there is a unique harmonic function $h : \{-N, \dots, N\} \to \mathbb{R}_+$ satisfying the boundary condition $h(-N) = h(N) = N^2$, it then follows that $g(x) = N^2$ for every $x \in \{-N, \dots, N\}$. Thus, $f(x) = N^2 - x^2$ for every $x \in \{-N, \dots, N\}$.