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Applied Stochastic Processes

Solution sheet 3

Solution 3.1

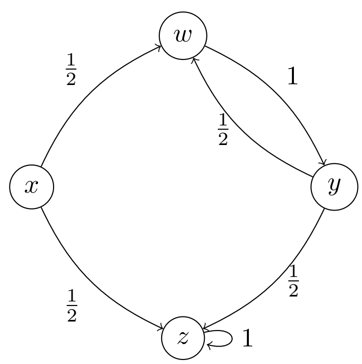
(a)
1. False.
2. False.
3. True. We have that:

$$\begin{aligned}\mathbf{P}_x(H_y < \infty,\, H_y > n \mid X_n = z) &= \mathbf{P}_x(H_y < \infty,\, \mid X_n = z, H_y > n)\mathbf{P}_x(H_y > n \,|X_n = z)\\ &= \mathbf{P}_z(H_y < \infty)\mathbf{P}_x(H_y > n \,|X_n = z)\end{aligned}$$

4. False.
5. False.

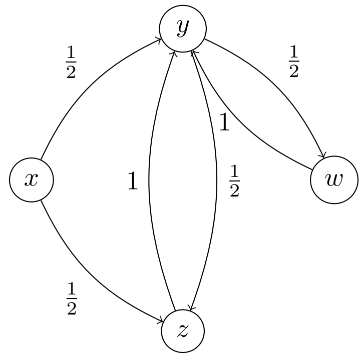
It is easy to find counterexamples for 1, 2, 4 and 5.

(b) Consider the following Markov Chain:



1. False. Using the Markov Chain above: $\mathbf{P}_x(T_1 = 2) = \frac{1}{2} \neq \frac{1}{4} = \mathbf{P}_x(T_2 = 2)$
2. False. Using the Markov Chain above: $\mathbf{P}_x(T_2 = 2) = \frac{1}{4} \neq \frac{1}{8} = \mathbf{P}_x(T_3 = 2)$
3. False. Using the Markov Chain above: $\mathbf{P}_y(T_1 = 2) = \frac{1}{2} \neq \frac{1}{4} = \mathbf{P}_y(T_2 = 2)$
4. False. We do not know which value the Markov Chain takes at T_2, even if it was finite.
5. False. Since T_1 or T_2 could be infinite, the expression $X_{T_1+T_2}$ need not be well-defined. However, the statement is true if both stopping times are finite almost surely.

(c) Consider the following Markov Chain:



1. False. Using the Markov Chain above: $\mathbf{P}_x(T_2 = T_3 = \infty) = \frac{1}{2} \cdot \frac{3}{4} \neq \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{11}{16} = \mathbf{P}_x(T_2 = \infty) \cdot \mathbf{P}_x(T_3 = \infty)$,
2. False. Using the Markov Chain above: $\mathbf{P}_x(T_1 = T_2 = \infty) = \frac{1}{2} \neq \frac{1}{2} \cdot \frac{3}{4} = \mathbf{P}_x(T_1 = \infty) \cdot \mathbf{P}_x(T_2 = \infty)$.
3. False. Using the Markov Chain above: $\mathbf{P}_y(T_1 = T_2 = \infty) = \frac{1}{2} \neq \frac{1}{2} \cdot \frac{3}{4} = \mathbf{P}_y(T_1 = \infty)\mathbf{P}_y(T_2 = \infty)$
4. True. We have that

$$\begin{aligned}\mathbf{P}_x(T_8 = \infty \mid T_7 < \infty) &= \sum_{k=0}^{\infty} \mathbf{P}_x(T_8 = \infty \mid T_7 = k)\mathbf{P}_x(T_7 = k \mid T_7 < \infty) \\ &= \sum_{k=0}^{\infty} \mathbf{P}_x(T_8 = \infty \mid T_7 = k, X_{T_1+\ldots+T_7} = y)\mathbf{P}_x(T_7 = k \mid T_7 < \infty) \\ &= \sum_{k=0}^{\infty} \mathbf{P}_y(H_y = \infty)\mathbf{P}_x(T_7 = k \mid T_7 < \infty) = \mathbf{P}_y(H_y = \infty)\end{aligned}$$

   Above we have used the Strong Markov Property with $f = \mathbb{1}_{T_8=\infty}$.
5. False. Using the Markov Chain above: $\mathbf{P}_x(T_8 = \infty \mid T_7 < \infty) = \frac{1}{2} \neq 1 = \mathbf{P}_x(H_x = \infty)$.

Note: Questions (b) 2, 3 and 5, and (c) 1, 2, 3 are true assuming that the inter-visit times are finite almost surely.

Solution 3.2

(a) By the definition of conditional expectation, we have to check that for all $A \in \mathcal{F}_k$:

$$\mathbf{E}_\mu\left[f\big((X_{k+n})_{n\geq 0}\big)\mathbb{1}_A\right] = \mathbf{E}_\mu\left[g(X_k)\mathbb{1}_A\right]$$

We have:

$$\begin{aligned}\mathbf{E}_\mu\left[f\big((X_{k+n})_{n\geq 0}\big)\mathbb{1}_A\right] &= \sum_{x\in S} \mathbf{E}_\mu\left[f\big((X_{k+n})_{n\geq 0}\big)\mathbb{1}_A | X_k = x\right]\mathbf{P}_\mu(X_k = x) \\ &= \sum_{x\in S} \mathbf{E}_x\left[f\big((X_n)_{n\geq 0}\big)\right]\mathbf{E}_\mu\left[\mathbb{1}_A | X_k = x\right]\mathbf{P}_\mu(X_k = x) \\ &= \sum_{x\in S} \mathbf{E}_\mu\left[g(x)\mathbb{1}_A | X_k = x\right]\mathbf{P}_\mu(X_k = x) = \mathbf{E}_\mu\left[g(X_k)\mathbb{1}_A\right]\end{aligned}$$

In the second equality we have used the Simple Markov Property, and in the third equality we have used the fact that $g(x)$ is a constant and thus we can move it inside the expectation.

(b) The solution is identical to (a) by changing k by T and applying the Strong Markov Property, thanks to T being finite almost surely.

Solution 3.3

(a) From the problem setup, we have that:

$$h(0) = P_0(\widetilde{H}_k < \widetilde{H}_0) = 0$$
$$h(k) = P_k(\widetilde{H}_k < \widetilde{H}_0) = 1$$

For $x \in \{1, \dots, k-1\}$, using the first-step analysis,

$$h(x) = P_x(\widetilde{H}_k < \widetilde{H}_0) = \sum_{y:|x-y|=1} P_x(\widetilde{H}_k < \widetilde{H}_0 \mid X_1 = y) P_x(X_1 = y).$$

From the transition probability we know that $P_x(X_1 = y) = \frac{1}{2}$. Thus,

$$h(x) = \frac{1}{2} h(x-1) + \frac{1}{2} h(x+1).$$

(b) Rewriting the recurrence equation, we notice that

$$h(x) - h(x-1) = h(x+1) - h(x).$$

Thus, the difference between consecutive terms is constantly d, and so $h(x) = dX + h(0)$. Using the boundary conditions we get that $d = \frac{1}{N}$, and thus $h(x) = \frac{x}{N}$.

Alternatively, we could find that the general solution to the recurrence $h(x) = 2h(x) - h(x-1)$ is of the form $A + Bx$, since its characteristic equation $r^2 - 2r + 1$ has a double root at $r = 1$. Solving for A and B yields the same result.

Solution 3.4 Let $x \in C$. We have $\mathbb{P}_x[\tau_C > kN] = \mathbb{P}_x[0 > kN] = 0$, and so the inequality holds true for all $k \geq 0$.

Let $x \in S \setminus C$. The inequality is trivial for $k = 0$. For $k \geq 1$, we prove it by induction over k. For $k = 1$, we have

$$\mathbb{P}_x[\tau_C > N] \leq \mathbb{P}_x[\tau_C > n(x)] = 1 - \mathbb{P}_x[\tau_C \leq n(x)] = 1 - \mathbb{P}_x[X_{n(x)} \in C] \leq 1 - \varepsilon. \tag{1}$$

For $k \geq 2$, it follows from the Markov property that

$$\begin{aligned}\mathbb{P}_x[\tau_C > kN] &= \sum_{y_1, \dots, y_{(k-1)N} \in S \setminus C} \mathbb{P}_x[\tau_C > kN, X_1 = y_1, \dots, X_{(k-1)N} = y_{(k-1)N}] \\ &= \sum_{y_1, \dots, y_{(k-1)N} \in S \setminus C} \mathbb{P}_{y_{(k-1)N}}[\tau_C > N] \cdot \mathbb{P}[X_1 = y_1, \dots, X_{(k-1)N} = y_{(k-1)N}].\end{aligned}$$

By (1), we have $P_{y_{(k-1)N}}[\tau_C > N] \leq 1 - \varepsilon$ for all $y_{(k-1)N} \in S$, and so

$$\begin{aligned}\mathbb{P}_x[\tau_C > kN] &\leq (1 - \varepsilon) \cdot \sum_{y_1, \dots, y_{(k-1)N} \in S \setminus C} \mathbb{P}_x[X_1 = y_1, \dots, X_{(k-1)N} = y_{(k-1)N}] \\ &= (1 - \varepsilon) \cdot \mathbb{P}_x[\tau_C > (k-1)N] \\ &\leq (1 - \varepsilon)^k,\end{aligned}$$

where we used the induction hypothesis $\mathbb{P}_x[\tau_C > (k-1)N] \leq (1-\varepsilon)^{k-1}$ in the last equation.

Solution 3.5

(a) We have

$$\mathbb{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbb{P}_0\left[H_a \leq n\right] = \mathbb{P}_0\left[X_n > a\right] + \mathbb{P}_0\left[H_a \leq n, X_n < a\right],$$

where we use that $a \geq 1$ is odd and $n \geq 0$ is even to ensure that $X_n \neq a$ $\mathbb{P}_0$-a.s.

(b) *Idea:* The law of $(X_{H_a+k})_{k\geq 0}$ is the same as the law of $(a - X_{H_a+k})_{k\geq 0}$, i.e. after hitting a at time H_a, we can reflect the trajectory of the SRW since steps $+1$ and -1 both occur with probability $1/2$.

More precisely, we have

$$\mathbb{P}_0\left[H_a \leq n, X_n < a\right] = \sum_{m=0}^{n} \mathbb{P}_0[X_n < a, H_a = m]. \tag{2}$$

By the strong Markov property,

$$\begin{aligned}\mathbb{P}_0[X_n < a, H_a = m] &= \mathbb{P}_0[X_n < a, H_a = m, H_a < \infty] \\ &= \underbrace{\mathbb{P}_0[X_{H_a+n-m} < a, H_a = m | H_a < \infty, X_{H_a} = a]}_{=\mathbb{P}_a[X_{n-m}<a]\cdot\mathbb{P}_0[H_a=m|H_a<\infty,X_{H_a}=a]} \cdot \mathbb{P}_0[H_a < \infty] \\ &= \mathbb{P}_a[X_{n-m} < a] \cdot \mathbb{P}_0[H_a = m].\end{aligned}$$

Since $\mathbb{P}_a[X_{n-m} > a] = \mathbb{P}_a[X_{n-m} < a]$ by symmetry, we deduce that

$$\begin{aligned}\mathbb{P}_0[X_n < a, H_a = m] &= \mathbb{P}_a[X_{n-m} < a] \cdot \mathbb{P}_0[H_a = m] \\ &= \mathbb{P}_a[X_{n-m} > a] \cdot \mathbb{P}_0[H_a = m] = \mathbb{P}_0[X_n > a, H_a = m],\end{aligned}$$

where we again used the strong Markov property in the last equality. Combined with (2), we conclude that

$$\begin{aligned}\mathbb{P}_0\left[H_a \leq n, X_n < a\right] &= \sum_{m=0}^{n} \mathbb{P}_0[X_n < a, H_a = m] \\ &= \sum_{m=0}^{n} \mathbb{P}_0[X_n > a, H_a = m] = \mathbb{P}_0[X_n > a, H_a \leq n] = \mathbb{P}_0[X_n > a].\end{aligned}$$

Finally, we deduce that

$$\begin{aligned}\mathbb{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] &= \mathbb{P}_0\left[X_n > a\right] + \mathbb{P}_0\left[H_a \leq n, X_n < a\right] \\ &= 2\mathbb{P}_0\left[X_n > a\right] = \mathbb{P}_0\left[X_n > a\right] + \mathbb{P}_0\left[X_n < -a\right] \\ &= \mathbb{P}_0[|X_n| > a].\end{aligned}$$

Solution 3.6

(a) Let us denote by $(X_n)_{n\geq 0}$ the Markov chain with transition probability corresponding to the rules of the game. Recall that $H_i = \inf\{n \geq 0; X_n = i\}$. Let us call $k_i = \mathrm{E}_i[H_9]$ for

$i \in \{1, \ldots, 9\}$. We observe that 9 is an absorbing state and that $k_9 = 0$. Then we can express H_9 as

$$H_9 = f((X_n)_{n\geq 0}) = \sum_{n=0}^{\infty} 1_{\{X_n<9\}}$$

where f is a measurable function. Then, for $i \in \{1, \ldots, 8\}$ we have $\mathbb{P}_i$-a.s. that

$$\begin{aligned}
k_i &= \sum_{j=1}^{9} \mathrm{E}_i[H_9|X_1 = j]\mathbb{P}_i[X_1 = j] \\
&= \sum_{j=1}^{9} \mathrm{E}_i[1_{\{X_0<9\}} + f((X_{n+1})_{n\geq 0})|X_1 = j]p_{i,j} \\
&\overset{(1)}{=} \sum_{j=1}^{9} (1 + \mathrm{E}_j[f((X_n)_{n\geq 0})])\, p_{i,j} \\
&= \sum_{j=1}^{9} (1 + k_j)p_{i,j}
\end{aligned}$$

where the equality (1) is justified by the Markov property. Applying this to the model, and considering the effect of the ladders and snakes we get to the following system of equations

$$\begin{aligned}
k_1 &= \frac{1}{2}(1 + k_7) + \frac{1}{2}(1 + k_5) \\
k_4 &= \frac{1}{2}(1 + k_5) + \frac{1}{2}(1 + k_1) \\
k_5 &= \frac{1}{2}(1 + k_1) + \frac{1}{2}(1 + k_7) \\
k_7 &= \frac{1}{2}(1 + k_4) + \frac{1}{2}(1 + k_9)
\end{aligned}$$

Since $k_9 = 0$ we can solve this system. We obtain that the average number of turns it takes to complete the game is given by $k_1 = 7$.

(b) Notice that the probability that a player starting from the middle square will complete the game without slipping to the square 1 is exactly $\mathbb{P}_5[H_9 < H_1]$. Using the Markov property repeatedly we get

$$\begin{aligned}
\mathbb{P}_5[H_9 < H_1] &= p_{5,6} \underbrace{\mathbb{P}_1[H_9 < H_1]}_{=0} + p_{5,7}\mathbb{P}_7[H_9 < H_1] \\
&= \frac{1}{2}(p_{7,8}\mathbb{P}_4[H_9 < H_1] + p_{7,9} \underbrace{\mathbb{P}_9[H_9 < H_1]}_{=1}) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_{4,5}\mathbb{P}_5[H_9 < H_1] + p_{4,6} \underbrace{\mathbb{P}_1[H_9 < H_1]}_{=0}) + \frac{1}{2}\right) \\
&= \frac{1}{8}\mathbb{P}_5[H_9 < H_1] + \frac{1}{4}
\end{aligned}$$

Then $\mathbb{P}_5[H_9 < H_1] = 2/7$.