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D-MATH
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Applied Stochastic Processes

Solution sheet 4

Solution 4.1

(a)
1. False. For the chain on $S = \{a, b\}$ with $p_x y = \delta_y^b$, we have that $H_x = \infty$ and $\widetilde{H}_x = 0$ $\mathbf{P}_x$-a.s.
2. False. For $x = y$, we note that $\widetilde{H_x} = 0$ $\mathbf{P}_y$-a.s. since $X_0 = y = x$. Thus, $\mathbf{P}_y[\widetilde{H_x} < \infty] = 1$, but we cannot say anything about $\mathbf{P}_y[H_x < \infty]$.
3. True. For $x \neq y$, we note that $\widetilde{H_x} \geq 1$ $\mathbf{P}_y$-a.s. since $X_0 = y \neq x$. Thus, $\widetilde{H_x} = H_x$ $\mathbf{P}_y$-a.s. and in particular,
$$\mathbf{P}_y[\widetilde{H_x} < \infty] = \mathbf{P}_y[H_x < \infty].$$
4. True. We note that $X_0 x$ $\mathbf{P}_x$-a.s. and thus,
$$\mathbf{E}_x[\widetilde{V_x}] = 1 + \mathbf{E}_x[V_x].$$
5. True. Under $\mathbf{P}_y$, $\mathbb{1}_{X_0 = x} = 0$ a.s.

(b)
1. True. The statement follows from the strong Markov property. More precisely, we will apply the strong Markov property for the stopping time $T = H_x$, the $\mathcal{F}_T$-measurable and bounded random variable $Z = 1$, and the measurable and bounded function $f_N : S^{\mathbb{N}} \to \mathbb{R}$ defined by
$$f_N(x_0, x_1, x_2, \ldots) = \left(\sum_{n=0}^{\infty} \mathbf{1}_{x_n = x}\right) \wedge N,$$
where we needed to introduce the truncation at N to ensure that f_N is bounded. Since $X_{H_x} = x$ and $H_x = \infty \implies V_x = 0$ $\mathbf{P}_y$-almost-surely, we obtain
$$\begin{aligned}\mathbf{E}_y[V_x \wedge N] &= \mathbf{E}_y\left[(V_x \wedge N) \cdot \mathbf{1}_{H_x < \infty}\right] \\ &= \mathbf{E}_y\left[V_x \wedge N | H_x < \infty, X_{H_x} = x\right] \cdot \mathbf{P}_y\left[H_x < \infty\right].\end{aligned}$$
Moreover,
$$\begin{aligned}\mathbf{E}_y\left[V_x \wedge N | H_x < \infty, X_{H_x} = x\right] &= \mathbf{E}_y\left[\left(\sum_{k \geq H_x} \mathbf{1}_{X_k = x}\right) \wedge N | H_x < \infty, X_{H_x} = x\right] \\ &= \mathbf{E}_y\left[f_N\left((X_{H_x + n})_{n \geq 0}\right) | H_x < \infty, X_{H_x} = x\right] \\ &= \mathbf{E}_x\left[f_N\left((X_n)_{n \geq 0}\right)\right] \\ &= \mathbf{E}_x\left[\widetilde{V_x} \wedge N\right],\end{aligned}$$
where we used the strong Markov property in the third equality. Taking the limit as $N \to \infty$, we conclude by monotone convergence that
$$\begin{aligned}\mathbf{E}_y[V_x] &= \lim_{N \to \infty} \mathbf{E}_y[V_x \wedge N] = \lim_{N \to \infty} \mathbf{E}_x\left[\widetilde{V_x} \wedge N\right] \cdot \mathbf{P}_y\left[H_x < \infty\right] \\ &= E_x\left[\widetilde{V_x}\right] \cdot \mathbf{P}_y\left[H_x < \infty\right] = (1 + \mathbf{E}_x[V_x]) \cdot \mathbf{P}_y\left[H_x < \infty\right],\end{aligned}$$
2. False. For a Markov Chain that is constant at $x = y$ $V_x^{(n)} + 1 = V_x^{(n+1)}$.

3. True. This follows from the Monotone Convergence Theorem.
4. False. This would imply $\mathbf{E}_x[V_x] = \infty$ but $\mathbf{P}_x[V_x = \infty] < 1$, which contradicts the Dichotomy Theorem.
5. True. For x recurrent, we have $\mathbf{P}_x[V_x = \infty] = 1$, so in order to have $\mathbf{P}_x[V_x = 2] > 0$, we need x to be transient. By the Lecture Notes,

$$\mathbf{P}_x[V_x = 2] = \mathbf{P}_x[V_x \geq 2] - \mathbf{P}_x[V_x \geq 3] = \rho_x^2 - \rho_x^3.$$

It is easy to verify that the function $f(\rho) = \rho^2 - \rho^3$ achieves its maximum on $[0, 1]$ at $\rho = 2/3$ with $f(2/3) = 4/27$. Consequently, it is possible to construct Markov chains with $\mathbf{P}_x[V_x = 2] = \rho$ if and only if $\rho \in [0, 4/27]$.
Concrete example for $\rho = 1/8$: We can consider the two-state Markov chain with state space $S = \{x, y\}$ and transition probability given by $p_{xx} = p_{xy} = 1/2$ and $p_{yy} = 1$. Then $\mathbf{P}_x[V_x = 2] = \mathbf{P}_x[X_1 = x, X_2 = x, X_3 = y] = p_{xx} \cdot p_{xx} \cdot p_{xy} = 1/8$.

Solution 4.2

(a) We prove recurrence of the state 0, but the same proof applies to all $x \in \mathbb{Z}$ (due to translation invariance of the transition probability of the SRW). We showed previously that for the SRW on $\mathbb{Z}$, i.e. the case of $\alpha = 1/2$ in the context of this exercise, we have for every $x \in \mathbb{Z}$ and for every $n \geq 0$,

$$p_{0x}^{(2n)} \leq p_{00}^{(2n)}.$$

Noting that $p_{0x}^{(2n)} = 0$ if $|x| > 2n$, we have

$$1 = \sum_{x \in \mathbb{Z}} p_{0x}^{(2n)} \leq (4n + 1) \cdot p_{00}^{(2n)},$$

which implies $p_{00}^{(2n)} \geq \frac{1}{4n+1}$ and thereby,

$$\mathbf{E}_0[V_0] = \sum_{n \geq 0} p_{00}^{(n)} = \infty.$$

The Dichotomy Theorem now implies that 0 is recurrent.

(b) Again, we only prove transience of the state 0. Moreover, it suffices to consider $\alpha > 1/2$ by symmetry (since $-X$ is a biased random walk on $\mathbb{Z}$ with parameter $1 - \alpha < 1/2$ and 0 is transient for X if and only if it is transient for $-X$).
First, we note that the Markov chain $X = (X_n)_{n \geq 0}$ under $\mathbf{P}_0$ has the same law as $X' = (X'_n)_{n \geq 0}$, defined as the partial sums $X'_0 = 0$ and $X'_n = \sum_{i=1}^n \xi_i$ of a collection $(\xi_i)_{i \geq 1}$ of i.i.d. random variables taking the values $+1$ (resp. -1) with probability α (resp. $1 - \alpha$). Therefore, by the strong law of large numbers,

$$\lim_{n \to \infty} \frac{X_n}{n} = \mathbf{E}_0[X_1] = 2\alpha - 1 > 0 \quad \mathbf{P}_0\text{-a.s.}.$$

Hence, for any $\epsilon \in (0, 2\alpha - 1)$,

$$\lim_{N \to \infty} \mathbf{P}_0[\bigcap_{n \geq N} \{X_n \geq \epsilon n\}] = \mathbf{P}_0[\bigcup_{N \geq 0} \bigcap_{n \geq N} \{X_n \geq \epsilon n\}] = 1,$$

and in particular, there exists $N \geq \epsilon^{-1}$ such that

$$1/2 \leq \mathbf{P}_0[\bigcap_{n \geq N} \{X_n \geq \epsilon n\}] \leq \mathbf{P}_0[\bigcap_{n \geq N} \{X_n \geq 1\}] \leq \mathbf{P}_0[\sum_{n \geq 1} \mathbf{1}_{X_n = 0} < N] \leq \mathbf{P}_0[V_0 < \infty].$$

By the Dichotomy Theorem, 0 is transient.

For parts (c)-(e), we construct an explicit coupling of reflected random walks and biased random walks for all values of $\alpha \in [0,1]$. On a general probability space $(\Omega, \mathcal{A}, \mathbf{P})$, let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. uniform random variables taking values in $[0,1]$. For every $\alpha \in [0,1]$, we define the following stochastic processes:

(i) $(X_0^{(\alpha)})_{n\geq 0}$ is defined by $X_0^{(\alpha)} = 0$ and for $n \geq 1$ by

$$X_n^{(\alpha)} = X_{n-1}^{(\alpha)} + \mathbb{1}_{U_n \leq \alpha} - \mathbb{1}_{U_n > \alpha}.$$

We note that $(X_0^{(\alpha)})_{n\geq 0}$ is a biased random walk on $\mathbb{Z}$ started at 0, i.e. it is a MC(δ^0,p) with the transition probability given on the exercise sheet.

(ii) $(Y_0^{(\alpha)})_{n\geq 0}$ is defined by $Y_0^{(\alpha)} = 0$ and for $n \geq 1$ by

$$Y_n^{(\alpha)} = \begin{cases} Y_{n-1}^{(\alpha)} + \mathbb{1}_{U_n \leq \alpha} - \mathbb{1}_{U_n > \alpha} & \text{if } Y_{n-1}^{(\alpha)} > 0, \\ Y_{n-1}^{(\alpha)} + 1 & \text{if } Y_{n-1}^{(\alpha)} = 0. \end{cases}$$

We note that $(Y_0^{(\alpha)})_{n\geq 0}$ is a reflected random walk on $\mathbb{N}$ started at 0, i.e. it is a MC(δ^0,p) with the transition probability given on the exercise sheet.

We define

$$H_0^{(\alpha,X)} := \inf\{n \geq 1 : X_n^{(\alpha)} = 0\} \quad \text{and} \quad H_0^{(\alpha,Y)} := \inf\{n \geq 1 : Y_n^{(\alpha)} = 0\},$$

and use analogous notation for $-X$ and $|X|$.

(c) Our goal is to show that $\mathbf{P}[H_0^{(\alpha,Y)} < \infty] = 1$ for $\alpha \leq 1/2$. We will use a comparison to a SRW ($\alpha = 1/2$). More precisely, we first note that for $\alpha \leq 1/2$,

$$H_0^{(\alpha,Y)} \leq H_0^{(1/2,Y)} \quad \mathbb{P}\text{-a.s.},$$

which follows from the definition of $Y^{(\alpha)}$. Moreover, $Y^{(1/2)}$ and $|X^{(1/2)}|$ have the same law. Thus, we conclude that

$$\mathbf{P}[H_0^{(\alpha,Y)} < \infty] \geq \mathbf{P}[H_0^{(1/2,Y)} < \infty] = \mathbf{P}[H_0^{(1/2,|X|)} < \infty] = \mathbf{P}[H_0^{(1/2,X)} < \infty] = 1,$$

where we used Exercise 3.2 (a) in the last equality.

(d) Our goal is to show that $\mathbb{E}[H_0^{(\alpha,Y)}] < \infty$ for $\alpha < 1/2$. We note that conditional on $\{X^{(\alpha)} = 1\}$, the processes $Y^{(\alpha)}$ and $X^{(\alpha)}$ take the same steps up to hitting 0 for the first time. Hence,

$$\begin{aligned}\mathbb{E}[H_0^{(\alpha,Y)}] =& \mathbb{E}[H_0^{(\alpha,X)} | X_1^{(\alpha)} = 1] = \sum_{k\geq 1} \mathbf{P}[H_0^{(\alpha,X)} \geq k | X_1^{(\alpha)} = 1] \\ &\leq \sum_{k\geq 1} \mathbf{P}[X_k^{(\alpha)} \geq 1 | X_1^{(\alpha)} = 1] = \sum_{k\geq 1} \mathbf{P}[\sum_{i=2}^{k} \xi_i \geq 0],\end{aligned}$$

where we set $\xi_i := \mathbb{1}_{U_i \leq \alpha} - \mathbb{1}_{U_i > \alpha}$ to be the steps of $X^{(\alpha)}$. By the law of large numbers, we already know that $\frac{1}{k-1}\sum_{i=2}^{k} \xi_i$ will be close to $\mathbb{E}[\xi_i] = 2\alpha - 1 < 0$ as $k \to \infty$. Quantitative upper bounds on $\mathbf{P}[\sum_{i=2}^{k} \xi_i \geq 0]$ follow from standard large deviation theory (see, e.g. Cramér's theorem). The proof is actually short, and so we include it here. For $\lambda \geq 0$, we obtain

$$\begin{aligned}\mathbf{P}[\sum_{i=2}^{k} \xi_i \geq 0] = \mathbf{P}[\exp(\lambda \sum_{i=2}^{k} \xi_i) \geq 1] \leq \mathbb{E}[\exp(\lambda \sum_{i=2}^{k} \xi_i)] = \prod_{i=2}^{k} \mathbb{E}[\exp(\lambda \xi_i)] \\ = \left(e^{\lambda} \cdot \alpha + e^{-\lambda} \cdot (1-\alpha)\right)^{(k-1)},\end{aligned}$$

where we used Markov's inequality and the independence of the $(\xi_i)_{i\geq 2}$. Minimizing $e^{\lambda}\cdot\alpha + e^{-\lambda}\cdot(1-\alpha)$ over $\lambda\in[0,\infty)$, we obtain that the minimum is attained at $\lambda = \ln(\sqrt{\frac{1-\alpha}{\alpha}}) > 0$ (since $\alpha < 1/2$). Hence,

$$\mathbf{P}[\sum_{i=2}^{k}\xi_i \geq 0] \leq \left(2\sqrt{\alpha(1-\alpha)}\right)^{(k-1)},$$

and so

$$\mathbb{E}[H_0^{(\alpha,Y)}] \leq \sum_{k\geq 1}\left(2\sqrt{\alpha(1-\alpha)}\right)^{(k-1)} = \frac{1}{1-2\sqrt{\alpha(1-\alpha)}} < \infty \quad \text{since } \alpha < 1/2.$$

(e) We first note that

$$\mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] = \mathbf{P}[H_0^{(1-\alpha,X)} < \infty | X_1 = -1] \geq \mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1],$$

where we used $\alpha \geq 1/2$ in the inequality. Second, by Exercise 3.2 (b), we know that 0 is transient for the biased random walk with $\alpha > 1/2$, i.e. we obtain

$$\begin{aligned} 1 > \mathbf{P}[H_0^{(\alpha,X)} < \infty] &= \alpha\cdot\mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] + (1-\alpha)\cdot\underbrace{\mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = -1]}_{\geq \mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1]} \\ &\geq \mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] \end{aligned}$$

By the definition of the processes $Y^{(\alpha)}$ and $X^{(\alpha)}$, we conclude that

$$\mathbf{P}[H_0^{(\alpha,Y)} < \infty] = \mathbf{P}[H_0^{(\alpha,Y)} < \infty | Y_1 = 1] = \mathbf{P}[H_0^{(\alpha,X)} < \infty | X_1 = 1] < 1.$$

Hence, 0 is transient for the reflected random walk with $\alpha > 1/2$.

Solution 4.3

Let $x \in V$. We observe that under $\mathbf{P}_x$, the process $Y = (Y_n)_{n\geq 1}$, defined by

$$Y_n := d(X_n, x),$$

is a reflected random walk on $\mathbb{N}$ starting at 0 with parameter

$$\alpha := \frac{d-1}{d} \geq 2/3,$$

i.e. a Markov chain $\mathrm{MC}(\delta^0, P')$ with transition probability given by $p'_{0,1} = 1$, $p'_{y,y+1} = \frac{d-1}{d}$, and $p'_{y,y-1} = \frac{1}{d}$ for $y \geq 1$. It now follows from Exercise 3.2 (e) that the state 0 is transient for Y. Clearly, the Markov chain X is in state x if and only if the Markov chain Y is in state 0. Therefore, the state x is transient for X.

Solution 4.4

(a) We have

$$\begin{aligned} \mathbf{P}_{(0,0)}[H_{(0,0)} < \infty] &= \sum_{y\in\mathbb{Z}^2, \mu(y)\neq 0} \mathbf{P}_{(0,0)}[H_{(0,0)} < \infty \mid X_1 = y]\mu(y) \\ &= \sum_{y\in\mathbb{Z}^2}\mu(y) = 1. \end{aligned}$$

Thus, $(0,0)$ is recurrent regardless of the choice of measure μ.

(b) The number of steps required for X to arrive at $y = (y_1, y_2)$ and then return to 0 is $||y||_1 + 1$. It follows that

$$\begin{aligned}\mathbf{E}_{(0,0)}[H_{(0,0)}] = \sum_{n\geq 1} n\mathbf{P}_{(0,0)}(H_{(0,0)} = n) = \sum_{n\geq 1} n\mathbf{P}_{(0,0)}\Big(\{y : ||y||_1 = n-1\}\Big) \\ = \sum_{y=(y_1,y_2)\in\mathbb{Z}^2} (||y||_1 + 1)\mu(y).\end{aligned}$$

Hence, $(0,0)$ is positive recurrent if $\sum_{y=(y_1,y_2)\in\mathbb{Z}^2}(||y||_1+1)\mu(y) < \infty$ and null recurrent if $\sum_{y=(y_1,y_2)\in\mathbb{Z}^2}(||y||_1+1)\mu(y) = \infty$.

(c)
1. Not a Markov chain in general. Denote $Y_n := ||X_n||_\infty$. Take $\mu(x) = \frac{1}{9}\mathbb{1}_{||x||_\infty\leq 1}$. Then $\mathbf{P}(Y_3 = 0 \mid Y_2 = 1, Y_1 = 1) = 1$ but $\mathbf{P}(Y_3 = 0 \mid Y_2 = 1) < 1$.
2. This is a Markov Chain in general. Here $S = \mathbb{N}$ and

$$p_{m,n} = \begin{cases} \sum_{x\in\mathbb{Z}^2:||x||_1=n} \mu(x) & \text{if } m = 0, \\ 1 & \text{if } m > 0 \text{ and } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

3. Not a Markov chain in general. Denote $Z_n := \Pi_x(X_n)$. Take $\mu(x) = \frac{1}{9}\mathbb{1}_{||x||_\infty\leq 1}$. Then $\mathbf{P}(Z_3 = 0 \mid Z_2 = 1, Z_1 = 1) = 1$ but $\mathbf{P}(Z_3 = 0 \mid Z_2 = 1) < 1$.