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Applied Stochastic Processes

Solution sheet 5

Solution 5.1

(a)
1. False. There are three communicating classes.
2. False. The state c is transient since d can be reached from c but c cannot be reached from d. Since the communicating class of b is $\{b, c\}$, the communicating class theorem implies that b is also transient.
3. True. By the previous argument we can see that the classes $\{a\}$ and $\{b, c\}$ are transient. Since the state space is finite, a result from the course gives that there exists at least one recurrent state, and so the class $\{d, e\}$ must be recurrent.
4. False. By definition, an irreducible Markov chain has exactly one communication class.
5. True. For example, take S to be countably infinite and define the transition probability P by $p_{xx} = 1$ for all $x \in S$. Then every state is recurrent and forms its own communication class.

(b)
1. False. The chain has two communicating classes for $\varepsilon = 0$.
2. False. For all $\varepsilon \geq 0$, $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is not reversible. It suffices to note that
$$\pi_c p_{cd} = \frac{1}{6} \neq \frac{1}{12} = \pi_d p_{dc}.$$
3. True. $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is reversible if and only if $\varepsilon = 0$. For $\varepsilon > 0$, we note that
$$\pi_b p_{bc} = \frac{1}{8} \cdot \varepsilon \neq \frac{1}{4} \cdot \varepsilon = \pi_c p_{cb}.$$
For $\varepsilon = 0$, we note that $\pi_a p_{ab} = \pi_b p_{ba} = \frac{1}{16}$ and $\pi_c p_{cd} = \pi_d p_{dc} = \frac{1}{6}$.
4. True. $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is stationary if and only if $\varepsilon = 0$. For $\varepsilon = 0$, it follows directly from (c) since reversible implies stationary. For $\varepsilon > 0$,
$$\pi_c = \frac{1}{4} \neq \frac{1}{4} - \frac{\varepsilon}{8} = \pi_b p_{bc} + \pi_c p_{cc} + \pi_d p_{dc}.$$
5. True. $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ is stationary for all $\varepsilon \geq 0$. To this end, it suffices to check that π is a left eigenvector associated to the eigenvalue 1 for the matrix
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} - \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \frac{1}{3} - \varepsilon & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Solution 5.2

(a) First, we note that $X_0 \sim \delta^1$. Second, we note that for all $n \geq 0$ and for all $k > 0$ and $\ell \in \mathbb{N}$, it follows from the definition of X_{n+1} that

$$\begin{aligned}\mathbb{P}[X_{n+1} = \ell | X_n = k] &= \mathbb{P}\left[\sum_{i=1}^{k} Z_i^{n+1} = \ell\right] \\ &= \sum_{z_1,\ldots,z_k \geq 0:\ z_1+\ldots+z_k=\ell} \mathbb{P}\left[Z_1 = z_1, \ldots, Z_k = z_k\right] \\ &= \sum_{z_1,\ldots,z_k \geq 0:\ z_1+\ldots+z_k=\ell} \nu(z_1) \cdot \ldots \cdot \nu(z_k) \\ &=: p_{kl}.\end{aligned}$$

For $k = 0$, we have $\mathbb{P}[X_{n+1} = 0 | X_n = 0] = 1 =: p_{00}$ for all $n \geq 1$. Moreover, by the independence of the $(Z_i^n)_{i,n\geq 1}$'s, we have for all $x_0, \ldots, x_{n+1}$,

$$\mathbb{P}[X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n] = \mathbb{P}[X_{n+1} = x_{n+1} | X_n = x_n],$$

whenever $\mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] > 0$. Consequently,

$$\mathbb{P}[X_0 = x_0, \ldots, X_{n+1} = x_{n+1}] = \delta^1_{x_0} \cdot p_{x_0 x_1} \cdot \ldots \cdot p_{x_n x_{n+1}},$$

and so we have shown that X is a Markov chain $\mathrm{MC}(\delta^1, P)$ with the transition probability $P = (p_{xy})_{x,y\in\mathbb{N}}$ defined above.

(b) $C_0 = \{0\}$ is a closed communication class and $C_1 = \{1, 2, \ldots\}$ is a communication class that is not closed. Since $C_0 \cup C_1 = E$, there are no other communication classes.

To see that $C_0 = \{0\}$ is a closed communication class, it suffices to note that $p_{00} = 1$. To see that $C_1 = \{1, 2, \ldots\}$ is a communication class that is not closed, we make the following observations: First, using $\nu(0), \nu(1) > 0$, we have $p_{i,i-1} > 0$ for all $i \geq 1$, and so $i \to i - 1$. Second, using $\nu(0) + \nu(1) < 1$, there exists $k \geq 2$ such that $\nu(k) > 0$, and so for all $i \geq 1$, $p_{i,ki} > 0$ and $i \to ki$. Combining these observations, $i \to j$ for all $i \geq 1$ and $j \geq 0$.

(c) C_0 is recurrent since $\mathbf{P}_0[H_0 = 1] = 1$. C_1 is transient, which follows from Corollary 2.14 in Section 2.10. More precisely, since for $i \geq 1$, $i \to 0$ but $0 \not\to i$, it follows that i is transient.

(d) If $\nu(0) = 0$ and $\nu(1) < 1$, then $X_{n+1} \geq X_n$ almost surely for all $n \geq 0$ and $X_{n+1} > X_n$ with positive probability. Consequently, there are infinitely many communication classes: $C_0 = \{0\}$, $C_1 = \{1\}$, $C_2 = \{2\}$, etc. As before, the class C_0 is recurrent and closed. The classes C_1, $C_2, \ldots$ are transient and not closed.

Solution 5.3

(a) Since X_n can take values in $\{0, \ldots, N\}$, we set $S := \{0, \ldots, N\}$.

On the one hand, for $x < N$, we set

$$p_{x,x+1} = 1 - \frac{x}{N},$$

as in order for X_n to grow by 1, the randomly selected particle must be from container B; this occurs with probability

$$\frac{\#\text{ of particles in } B}{\#\text{ of total particles}} = \frac{N - x}{N}.$$

On the other hand, for $x > 0$, the only other option is for the amount of particles in A to decrease by 1, which happens with probability $\frac{x}{N}$, and so

$$p_{x,x-1} = \frac{x}{N}.$$

Whenever $|x-y| \neq 1$ for $x, y \in S$, we set $p_{xy} = 0$. It can easily be verified that $P = (p_{xy})_{x,y\in S}$ defines a trasition probability.

(b) Our goal is to identify a stationary distribution; this would represent the equilibrium distribution of particles. To this end, we try to find a reversible distribution π. By Proposition 3.1, we know that it would also be stationary.

By definition of reversibility, π needs to satisfy for all $x \in \{0, \dots, N-1\}$,

$$\pi_x p_{x,x+1} = \pi_{x+1} p_{x+1,x}.$$

We use this to calculate π_x explicitly and see if this defines a proper distribution.

$$\pi_{x+1} = \frac{\pi_x(1-\frac{x}{N})}{\frac{x+1}{N}} = \pi_x \frac{N-x}{x+1} \overset{\text{(Induction)}}{=} \pi_0 \frac{N\cdots(N-x)}{(x+1)!}. \tag{1}$$

Thus we find that $\pi_x = \binom{N}{x}\pi_0$. Since π should define a distribution, we must have $\sum_{x\in S}\pi_x = 1$. Hence we find

$$\pi_0 = \left(\sum_{x\in S}\binom{N}{x}\right)^{-1} = \frac{1}{2^N}. \tag{2}$$

Hence,

$$\pi_x = \binom{N}{x}\frac{1}{2^N},$$

the binomial distribution; which is (as we have shown) reversible, and thus stationary.

Solution 5.4

First, if π is reversible, then it follows directly from Proposition 3.1 that it is stationary.

Second, we assume that π is stationary and aim to prove that it is reversible. In the case $p = q = 1$, every distribution π is trivially reversible and stationary. In the case $p \neq 1$ or $q \neq 1$, we deduce from $\pi P = \pi$ that

$$\pi_1(1-p) = \pi_2(1-q).$$

Plugging in $\pi_2 = 1 - \pi_1$, we get

$$1 - q = \pi_1(2-p-q) \quad \Longleftrightarrow \quad \pi_1 = \frac{1-q}{2-p-q}.$$

Hence,

$$(\pi_1, \pi_2) = \left(\frac{1-q}{2-p-q}, \frac{1-p}{2-p-q}\right),$$

which is reversible since $\pi_1 p_{12} = \pi_2 p_{21}$.

Solution 5.5

(a) We check that the two properties in the definition of transition probability are satisfied. For every $x, y \in S$, the definition directly implies $\hat{p}_{xy} \geq 0$. For every $x \in S$ with $\pi(x) = 0$,

$$\sum_{y \in S} \hat{p}_{xy} = \sum_{y \in S} \mathbf{1}_{x=y} = 1,$$

and for every $x \in S$ with $\pi(x) > 0$,

$$\sum_{y \in S} \hat{p}_{xy} = \sum_{y \in S} \frac{\pi(y) p_{yx}}{\pi(x)} = \frac{\pi(x)}{\pi(x)} = 1$$

by stationarity of π.

(b) Let $f, g \in \mathrm{L}^\infty(S)$.

$$\begin{aligned}\langle Pf, g \rangle_\pi &= \sum_{x \in S} (Pf)(x)\, g(x)\, \pi(x) = \sum_{x,y \in S} \pi(x)\, p_{xy}\, f(y)\, g(x) \\ &= \sum_{x,y \in S} \pi(y)\, \hat{p}_{yx}\, f(y)\, g(x) = \sum_{y \in S} f(y)\, (\hat{P}g)(y)\, \pi(y) = \langle f, \hat{P}g \rangle_\pi\end{aligned}$$

In the third equality, we have used that the definition of $\hat{P}$ implies $\pi(x)p_{xy} = \pi(y)\hat{p}_{yx}$ for all $x, y \in S$. If $\pi(y) > 0$, this is clear. If $\pi(y) = 0$, we must have $\pi(x) = 0$ or $p_{xy} = 0$ by stationarity of π.

(c) If π is reversible, then $\pi(y)\hat{p}_{yx} = \pi(x)p_{xy} = \pi(y)p_{yx}$, and so the same calculation as in part (b) shows that $\langle Pf, g \rangle_\pi = \langle f, Pg \rangle_\pi$, i.e. P is self-adjoint.

Fix arbitrary $x, y \in S$ and set $f := \delta_x$ and $g := \delta_y$. If P is self-adjoint, then

$$\pi(y)p_{yx} = \langle Pf, g \rangle_\pi = \langle f, Pg \rangle_\pi = \pi(x)p_{xy},$$

and so p is reversible.