Applied Stochastic Processes

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ETH zürich

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Plan for Today

1. **What is a stochastic process?**

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2. **Goals/content of the lectures**

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3. **Administrative and practical information**

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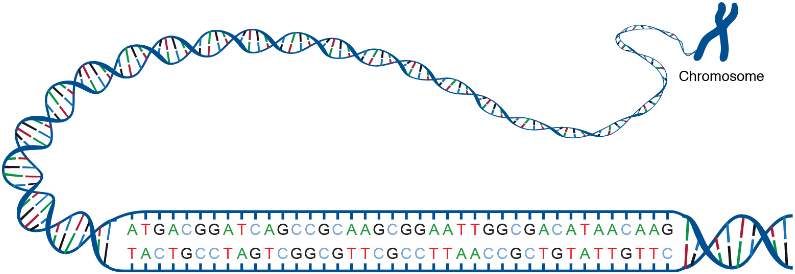
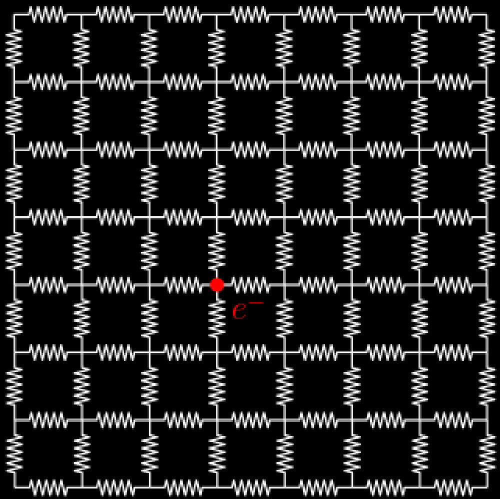
3. **Administrative and practical information**
4. **Chapter 1: Markov Chains**

1. What is a stochastic process?

A stochastic process in Genetics



In Electronics



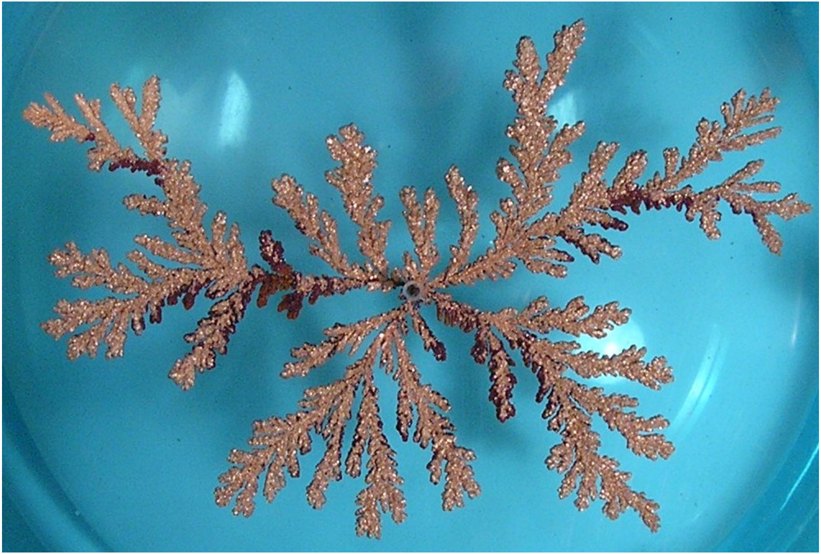
On the road

In Finance


ufstieg des
Euro, gemessen in Franken
1.15
1.10
Neue Zürcher Zeitung

In Geneva

Mysterious



Mathematical definition

Setup:

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space.
- $(S, \mathcal{S})$ measured space. "state space"

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A random variable in S is a measurable map

$$X : \Omega \to S.$$

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$$\mathbb{P}[X = -1] = \mathbb{P}[X = 1] = \frac{1}{2}. \qquad \text{"Coin Flip".}$$

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- $S = \mathbb{R}$, $X \sim \mathcal{N}(0, 1)$.

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A discrete-time stochastic process in S is a sequence $X = (X_n)_{n \in \mathbb{N}}$ of random variables in S.

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Examples:

- $(X_n)_{n \in \mathbb{N}}$ iid coinflips.
- $(S_n)_{n \in \mathbb{N}}$ where $S_n = X_1 + \cdots + X_n$. " random walk"
- $(M_n)_{n \in \mathbb{N}}$ martingale.

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Definition

A continuous-time stochastic process in S is a collection $X = (X_t)_{t \in \mathbb{R}_+}$ of random variables in S.

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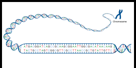
Definition

A continuous-time stochastic process in S is a collection $X = (X_t)_{t \in \mathbb{R}_+}$ of random variables in S.

Continuous Stochastic Process = "random function".

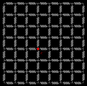
Applications



$X_n \in \underbrace{\{A,C,T,G\}}_{E}$

$(X_n)_{n\in\mathbb{N}}$

"MARKOV CHAIN"



$X_n \in \mathbb{Z}^2$

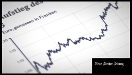
$(X_n)_{n\in\mathbb{N}}$

"RANDOM WALK"

POISSON PROCESS

$(N_t)_{t\in\mathbb{R}_+}$

N_t = "number of cars passing a fixed place during $[0,t)$"



$B_t \in \mathbb{R}$

$(B_t)_{t\in\mathbb{R}_+}$

Brownian motion.

$\Gamma_t \in \mathbb{R}^2$

$(\Gamma_t)_{t\in\mathbb{R}_+}$

SLE process

$(K_t) \in \mathcal{K}(\mathbb{R}^2)$

$(K_t)_{t\in\mathbb{R}_+}$

DLA.

2. Goals/content of the lectures

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1. Markov Chains

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2. Renewal Processes

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3. Poisson Point Processes

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- Prepare to Brownian Motion.