Coordinator
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Brownian Motion and Stochastic Calculus

Exercise Sheet 1

Submit by 12:00 on Wednesday, February 26 via the course homepage.

Exercise 1.1 *(Monotone class theorem)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y : \Omega \to \mathbb{R}$ random variables. Use the monotone class theorem to show that a random variable Z is $\sigma(X, Y)$-measurable if and only if it is of the form $Z = f(X, Y)$ for some (Borel)-measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

Hint: It may be helpful to start by assuming that Z is bounded.

Exercise 1.2 *(Modifications and indistinguishability)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t \geqslant 0}$ and $Y = (Y_t)_{t \geqslant 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Recall that two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P[Z_t = Z'_t] = 1$ for each $t \geqslant 0$, while Z and Z' are *indistinguishable* if $P[Z_t = Z'_t \text{ for all } t \geqslant 0] = 1$.

(a) Assume that X and Y are both P-a.s. right-continuous or both P-a.s. left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

(b) Give an example showing that one of the implications in part (a) does not hold for general stochastic processes X and Y.

Exercise 1.3 *(Measurability of stochastic processes)* Let $X = (X_t)_{t \geqslant 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, P)$. The aim of this exercise is to show the following chain of implications:

X optional $\Rightarrow$ X progressively measurable $\Rightarrow$ X product-measurable and adapted.

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.

Remark: The same conclusion holds true if every path of X is left-continuous.

Hint: For fixed $t \geqslant 0$, consider the approximating sequence of processes Y^n on $\Omega \times [0, t]$ given by $Y_0^n = X_0$ and $Y_u^n = \sum_{k=0}^{2^n - 1} \mathbf{1}_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$.

(c) Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.

Exercise 1.4 *(Transformations of Brownian motion)* Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion.

(a) Show that $(-W_t)_{t \geqslant 0}$ is a Brownian motion.

(b) Show that for any $c \neq 0$, $(cW_{t/c^2})_{t \geqslant 0}$ is also a Brownian motion.